NONNEGATIVITY CRITERION FOR A DEGENERATE QUADRATIC FORM WITH TWO-DIMENSIONAL CONTROL

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As is known, the quadratic conditions for a local minimum in problems of the classical calculus of variations (and also in certain optimal control problems) lead to the study of definiteness of an integral quadratic functional, and the main assumption in this study is the strengthened Legendre condition.

The case where the Legendre condition degenerates at least at one point has been studied much less (note here [4, 8, 1]). Meanwhile, it is of interest both from the theory of quadratic forms themselves and because it can be well realized in particular variational problems. If we restrict ourselves to the degeneration of the Legendre condition at a single point, then the first interesting and nontrivial functional has the following form:

$$J = \int_0^1 \big(t^2(u,u) - 2bt(Px,u) + (Dx,x)\big)\, dt, \tag{1}$$

where

$$\dot{x} = u, \qquad x(1) = 0. \tag{2}$$

Here, x and u are two-dimensional, P is the matrix of rotation by 90°, $b \in \mathbf{R}$ is an arbitrary parameter, and D is a constant symmetric matrix. The function $u(t)$ is assumed to belong to $L_\infty[0,1]$ for a while, i.e., $x(t)$ is Lipschitzian (we have chosen the minus sign for the middle term of (1) for the convenience of further formulas).

The form of this functional is obtained from the following considerations: if for some functional with the term $(R(t)u,u)$, the Legendre condition is fulfilled, i.e., if $R(t) \ge 0$, but it degenerates at a single point t_0, then a typical degeneration has the form $R = (t-t_0)^2$. The coefficient of (Px,u) also must degenerate at t_0 (otherwise, this term becomes leading, and J has negative values in advance); moreover, this degeneration must be of first order (if the degeneration is higher, the mixed term is smaller in order than both extreme terms). It is natural to begin the study of this functional with variations of $x(t)$ concentrated in a neighborhood of this point, i.e., under the conditions $x(t_0-\varepsilon) = x(t_0+\varepsilon) = 0$, and then, restricting ourselves (due to symmetry) to the consideration of the half-interval $[t_0, t_0+\varepsilon]$, we arrive at the form (1), (2).

We call attention to the fact that functional (1) has the self-similarity property (or, as one can also say, the "fractality" property): its nonnegativity does not depend on the closed interval $[0,T]$ on which it is considered (under the condition $x(T)=0$ on its right end point); therefore, we set $T=1$.

We pose the following question: for which parameters b and D is the functional J nonnegative on the above set of functions? (Then, obviously, it will also be nonnegative on its natural closure, whose description will be given below.) We cannot directly apply the classical Jacobi condition because the strengthened Legendre condition is not fulfilled (it degenerates at the point $t=0$).

1. We first establish that the left end of $x(t)$ can also be assumed to be zero.

Lemma 1. *The property $J(x)\ge 0$ for all Lipschitzian $x(t)$ with the boundary condition $x(1)=0$ is equivalent to the property $J(x)\ge 0$ for all Lipschitzian $x(t)$ with the boundary conditions $x(0)=x(1)=0$.*

Proof. It suffices to prove that if $J(\hat{x})<0$ for a certain Lipschtzian $\hat{x}$ with the condition $\hat{x}(1)=0$, then there exists a Lipschitzian x with the condition $x(0)=x(1)=0$ for which $J(x)<0$ as well.

Without loss of generality, we can assume that $\hat{x}=\text{const}$, i.e., $\hat{u}=0$ on the closed interval $[0,\Delta]$ for a certain $\Delta>0$ (since the set of such u is everywhere dense in $L_\infty[0,1]$ with respect to any integral metric). Take any $\varepsilon<\Delta$ and construct a function $x(t)$ which linearly grows from 0 up to $\hat{x}(\varepsilon)$ on $[0,\varepsilon]$ and then coincides with $\hat{x}(t)$. Thus, the constructed x and u differ from the initial $\hat{x}$ and $\hat{u}$ only on the interval $[0,\varepsilon]$ on which $|u(t)|\simeq 1/\varepsilon$ (a quantity of order $1/\varepsilon$), and, therefore, $|tu|\le\text{const}$ and also $t^2\,|u|^2\le\text{const}$. Therefore, $|J(x)-J(\hat{x})|\le\text{const}\cdot\varepsilon$, and then, for a small ε , we obtain $J(x)<0$; this is what was required to be proved. □

In principle, this lemma makes it possible to apply the Jacobi conditions, since we now can assume that $x(0)=0$, and then, according to the general idea of these conditions, we can move the left end-point of this closed interval. For $\theta>0$, the strengthened Legendre condition is fulfilled on each closed interval $[\theta,1]$, and, therefore, we can seek out a point t_* conjugate to the point $t=1$. If there is no such point on the interval (0,1), then for any $\theta>0$, the functional $J\ge 0$ on $[\theta,1]$, and then, by continuity, $J\ge 0$ on $[0,1]$ as well. (The same procedure can also be performed for the free $x(0)$, without using Lemma 1, but in this case, in considering J on $[0,1]$, we need to add to it the integrated term $(Dx(\theta),x(\theta))\cdot\theta$ corresponding to the integral over $[0,\theta]$, and then the coefficients of the functional are no longer constant.)

But we proceed in another way. Taking an arbitrary constant symmetric matrix S, we add the expression

$$-\frac{d}{dt}\,[t(Sx,x)]=-2t(Sx,u)-(Sx,x)$$

under the integral sign in (1); obviously, this expression does not change the value of the functional (since the integral of it is $(t(Sx,x))|_0^T = 0)$. Then the integrand becomes

$$[tu - (S+bP)x]^2 + (Mx,x),$$

where

$$M = D - S - S^2 - b^2E + b(PS - SP)$$

and E is the identity matrix. If, for a certain S, we obtain $M \geq 0$, then the nonnegativity of J is obvious. If we cannot find such an S, then we intend to show that $J(x) < 0$ for a certain x.

2. Before implementing this program, we note that functional (1) can be rewritten in the following two interesting ways. Under the change $t = e^{-\tau}$, $\dfrac{dt}{t} = -d\tau$, and, respectively, $tu = t\dfrac{dx}{dt} = -\dfrac{dx}{d\tau} = -w(\tau)$, the above functional transforms into the functional

$$J = \int_0^\infty e^{-\tau}\left[w^2 + 2b(Px,w) + (Dx,x)\right]\,d\tau, \tag{3}$$

$$\frac{dx}{d\tau} = w, \qquad x(0) = 0.$$

Such functionals are typical for mathematical economics models; the coefficient $e^{-\tau}$ is called the discounting factor. The minus sign in the middle term of (1) was chosen so that functional (3) has the "canonical" form with the plus sign.

If we now set $e^{-\tau/2}x = z$ and $e^{-\tau/2}w = v$, then we obtain the following linear system with constant coefficients:

$$\frac{dz}{d\tau} = -\frac{1}{2}\,z + v, \qquad z(0) = 0, \tag{4}$$

and the functional, which also has constant coefficients:

$$J = \int_0^\infty \left[v^2 + 2b(Pz,v) + (Dz,z)\right]\,d\tau. \tag{5}$$

In order not to study the convergence of the integral, we examine this functional on all compactly supported $z(t)$, i.e., such that $z(t) = 0$ for all sufficiently large t; here, by Eq. (4), $v(t)$ is also compactly supported. (For functional (1), this corresponds to the consideration of only those x for which $x(t) = 0$ on $[0,\varepsilon]$ for certain $\varepsilon > 0$.)

The differential constraint (4) can be simplified by setting $\dfrac{dz}{d\tau} = u = -\frac{1}{2}\,z + v$, expressing $v = u + \frac{1}{2}\,z$ from this, and substituting the result in (5). Here, $(Pz,z) = 0$, and the terms of the form $\int 2(z,\dot z)\,dt = (z,z)|_0^\infty = 0$ vanish. Thus, changing the notation τ, z to the usual t, x, we obtain

$$\dot x = u, \qquad x(0) = 0, \tag{6}$$

$$J = \int_0^\infty \left[u^2 + 2b(Px, u) + (Qx, x) \right] \, dt, \tag{7}$$

where $Q = D + \frac{1}{4}E$. Our functional will be studied precisely in this form. We first establish a property which is true for functional (6), (7) in the space $\mathbf{R}^n$ of any dimension.

Lemma 2. *If* $J \geq 0$, *then* $Q \geq 0$ (*the Legendre condition*) *.*

Proof. Assume the contrary: let there exist $h \in \mathbf{R}^n$ such that $(Qh, h) < 0$. We set $x(t) = h$ and $u(t) = 0$ on the closed interval $[1, T]$, and on the closed intervals $[0, 1]$ and $[T, T+1]$ let x vary linearly from 0 to h and from h to 0, respectively. Since the integral over $[1, T]$ is the negative quantity $(T-1)(Qh, h)$ of order T and the integrals over intervals $[0, 1]$ and $[T, T+1]$ are finite, we obtain $J(x) < 0$ for large T, i.e., arrive at a contradiction. □

3. Following the above idea, we take an arbitrary symmetric matrix S and add the expression $\dfrac{d}{dt}(Sx, x) = 2(Sx, u)$ under the integral sign in order to extract a complete square from the terms containing u. Then we obtain

$$J = \int_0^\infty \big([u + (S + bP)x]^2 + (Mx, x) \big) \, dt, \tag{8}$$

where

$$(Mx, x) = (Qx, x) - (Sx + bPx)^2,$$

i.e.,

$$M = Q - (S + bP)^*(S + bP) = Q - S^2 + b(PS - SP) - b^2 E. \tag{9}$$

If we obtain $M \geq 0$ under this procedure, then, obviously, $J \geq 0$ on all compactly supported functions satisfying (6).

We ask the following question: for which b and Q can one obtain the inequality $M \geq 0$ by choosing an appropriate matrix S ?

Without loss of generality, we can assume that the matrix Q is diagonal (because under the rotation of the two-dimensional vector x and the corresponding vector u, the quadratic forms (u, u) and (Px, u) do not change), i.e.,

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix},$$

where, according to Lemma 2, $q_1 \geq 0$ and $q_2 \geq 0$. We seek S in the form

$$S = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

where c is an unknown parameter for now. (We will see below that the consideration of such an S is sufficient for our purposes.) Then, carrying out simple calculations, we obtain

$$M = Q - c^2E - b^2E + 2bc \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 - (b+c)^2 & 0 \\ 0 & q_2 - (b-c)^2 \end{pmatrix}, \tag{10}$$

and the question reduces to the following: for which b, q_1, and q_2 does there exist c such that

$$q_1 - (b+c)^2 \geq 0 \qquad \text{and} \qquad q_2 - (b-c)^2 \geq 0\ ?$$

The latter is equivalent to the property that, for a certain c,

$$\sqrt{q_1} \geq |b+c| \qquad \text{and} \qquad \sqrt{q_2} \geq |b-c|. \tag{11}$$

It is easy to see that such a c can be found if and only if

$$\frac{1}{2}\,(\sqrt{q_1} + \sqrt{q_2}) \geq |b|. \tag{12}$$

Thus, we have established that if (12) is fulfilled, then there exists a symmetric matrix S (of the indicated form) such that the corresponding $M \geq 0$, and, therefore, $J \geq 0$.

What happens if (12) is not fulfilled? There is no matrix S of the required form (with zero diagonal) but does this mean that the property $J \geq 0$ is violated? We will show that it is really so, and this is a key point of the approach proposed here.

Without loss of generality, we assume for convenience that $b \geq 0$ (otherwise, we make the change $x_1 \leftrightarrow x_2$, $u_1 \leftrightarrow u_2$ under which (Px, u) transforms into $-(Px, u)$, and all other terms in the expression for J do not change).

Thus, let

$$\frac{1}{2}\,(\sqrt{q_1} + \sqrt{q_2}) < b.$$

In this case, obviously, there exists c such that

$$\sqrt{q_1} < b + c, \qquad \sqrt{q_2} < b - c \tag{13}$$

(for example, we can take $c = \frac{1}{2}\,(\sqrt{q_1} - \sqrt{q_2})$), i.e.,

$$q_1^2 < (b+c)^2, \qquad q_2^2 < (b-c)^2;$$

both entries of the matrix M are negative, and, therefore, $M < 0$ (is negative-definite). Then, for certain $\delta > 0$, we have $(Mx, x) \leq -\delta |x|^2 \quad \forall x$, i.e., the second term under the integral sign in (8) is negative-definite.

We now try to find an admissible pair (x, u) for which the first term under the integral sign in (8) is zero, i.e., we set $u + (S + bP)x = 0$. Then we obtain the equation

$$\dot{x} = -(S + bP)x. \tag{14}$$

Lemma 3. *If* (13) *is fulfilled, then Eq.* (14) *has a periodic solution of the form* $x = f \sin\omega t + h\cos\omega t$ *for a certain* $\omega \neq 0$ *and noncollinear vectors* f, $h \in \mathbf{R}^2$.

Proof. It is sufficient to show that the matrix $R = (S + bP)$ has a purely imaginary eigenvalue $\lambda = i\omega \neq 0$. Since

$$R = \begin{pmatrix} 0 & c-b \\ c+b & 0 \end{pmatrix},$$

the equation for eigenvalues has the form $\lambda^2 - (c^2 - b^2) = 0$. If $c^2 - b^2 < 0$, i.e., if $|c| < b$, then this equation has two purely imaginary roots $\lambda = \pm i\omega$, $\omega = \sqrt{b^2 - c^2} > 0$. Let us show that precisely this case is realized. It follows from inequalities (13) that

$$-c < d - \sqrt{q_1} \leq b, \qquad c < b - \sqrt{q_2} \leq b,$$

i.e., $\pm c < b$, which means that $|c| < b$. The lemma is proved. □

Since the vectors f and h are linearly independent, $x(t) = f \sin \omega t + h \cos \omega t$ describes an ellipse in $\mathbf{R}^2$, and we can assume that $|x| \geq 1$ on it. Now, consider the indicated solution $x(t)$ on a large time interval $[1, T]$ (as was earlier done for $x = \text{const}$). On this interval, the first term (the square) in expression (8) is equal to zero by definition, and since $|x| \geq 1$, we have $(Mx, x) \leq -\delta$; therefore, the integral over the interval $[1, T]$ is a negative quantity $\leq -\delta(T-1)$ of order T. On the closed intervals [0,1] and $[T, T+1]$, as before, we reduce x to zero end-values at the points 0 and $T+1$; since $x(t)$ is bounded, the integrals over these intervals also make only a finite contribution to the functional. Therefore, on the entire closed interval $[0, T+1]$, for large T we obtain a function $\hat{x}(t)$ for which $J(\hat{x}) \leq -\delta T/2 < 0$.

Remark 1. We stress once more that here, a key point consists of the property that the found cyclic solution of Eq. (14) can be "rolled up" for an arbitrarily long time, thereby accumulating an arbitrarily large negative integral of (Mx, x) and preserving the bounded value of x. Moreover, the first term in (8) remains equal to zero all the time by virtue of (14).

Thus, we have shown that if inequality (11) is not fulfilled, then there exists a compactly supported function $\hat{x}(t)$ for which $J(\hat{x}) < 0$. Therefore, taking into account what was said above, we have established the following property.

Theorem 1. *Functional* (7) *is nonnegative on all compactly supported functions satisfying Eq.* (6) *if and only if the eigenvalues of the matrix* Q *are nonnegative and satisfy inequality* (12).

Remark 2. Squaring (12), we obtain the equivalent inequality $(q_1 + q_2) + 2\sqrt{q_1 q_2} \geq 4b^2$, which can be written in terms of the original matrix Q not reducing it to the diagonal form, i.e.,

$$\text{Tr } Q + 2\sqrt{\det Q} \geq 4b^2. \tag{15}$$

4. Particular cases. (a) Let $b = 0$ and $Q = 0$. Then the equality is fulfilled in (12), and by Theorem 1, we have $J = \int_0^\infty u^2\,dt \geq 0$ for any $x(t)$ such that $\dot{x} = u$ and $x(0) = 0$. Certainly, this result is obvious without Theorem 1 as well.

Let us examine what the obtained conditions mean for functionals (1) and (3). Since $Q = D + \frac{1}{4}E$, the inequality $Q \geq 0$ means that $D + \frac{1}{4}E \geq 0$, and inequality (12) means that

$$\frac{1}{2}\left(\sqrt{d_1 + \frac{1}{4}} + \sqrt{d_2 + \frac{1}{4}}\right) \geq |b|. \tag{16}$$

(b) Consider the case $b = 0$ and $D = -\frac{1}{4}E$ (i.e., $Q = 0$) . Then functional (3) has the form (we write t instead of τ once again)

$$J = \int\limits_0^\infty e^{-t}\left(u^2 - \frac{1}{4}x^2\right)\,dt \geq 0 \tag{17}$$

on all compactly supported $x(t)$ such that

$$\dot{x} = u, \qquad x(0) = 0. \tag{18}$$

In other words, the following inequality is fulfilled for such functions:

$$\int\limits_0^\infty e^{-t}x^2\,dt \;\leq\; 4\int\limits_0^\infty e^{-t}u^2\,dt. \tag{19}$$

If we introduce the Hilbert space $H = L_2[0,\infty)$ with weight e^{-t}, then (because all compactly supported functions are dense in this space) it follows from (19) that the integral operator $u \mapsto x$ given by formula (18) is a linear bounded operator $H \to H$, and its norm does not exceed $\sqrt{4} = 2$. Actually, its norm is equal to 2, since the constant 4 in inequality (19) is sharp.

(c) For functional (5), this property means that for any compactly supported function z satisfying (4), we have

$$J = \int\limits_0^\infty \left(v^2 - \frac{1}{4}z^2\right)\,dt \geq 0,$$

i.e.,

$$\int\limits_0^\infty z^2\,dt \;\leq\; 4\int\limits_0^\infty v^2\,dt. \tag{20}$$

This implies that for any function $v \in L_2[0,\infty)$, the function $z(t)$ satisfying the equation

$$\dot{z} = -\frac{1}{2}\,z + v, \qquad z(0) = 0, \tag{21}$$

also belongs to $L_2[0,\infty)$, and, moreover, the norm of the operator $v \mapsto z$ does not exceed (and actually is equal to) 2.

Also, we note that the same property holds for the equation

$$\dot{z} = -kz + v, \qquad z(0) = 0 \tag{22}$$

for any $k > 0$. (It reduces to (21) by simple scaling.) Namely, the following lemma is true.

Lemma 4. *For any function $v \in L_2[0,\infty)$, the function $z(t)$, which is a solution of Eq. (22), also belongs to $L_2[0,\infty)$, and, moreover, the norm of the operator $v \mapsto z$ is equal to $1/k$.*

Remark 3. The above operator, although being integral and, therefore, completely continuous on the space $L_2[0,T]$ for any finite T, is not completely continuous on the space $L_2[0,\infty)$. In particular, it has a purely continuous spectrum. (On the complex plane, it is the disk whose diameter is the segment $[0, 1/k]$ of the real axis.)

(d) For functional (1) with $D = -\frac{1}{4}E$ and $b = 0$, we obtain the inequality

$$J = \int_0^1 \left(t^2 u^2 - \frac{1}{4} x^2 \right) dt \ge 0,$$

i.e.,

$$\int_0^1 x^2\, dt \;\le\; 4 \int_0^1 t^2 u^2\, dt \tag{23}$$

for all x, u satisfying Eq. (2) that are equal to zero in a certain neighborhood of $t = 0$. Then the same inequality is also true for all $u(t)$ for which the integral on the right-hand side of (23) converges (i.e., for all $u(t)$ from the space $L_2[0,1]$ with weight t^2); moreover, the integral on the left-hand side also converges by virtue of estimate (23). Note that in all three inequalities (19), (20), and (23), the dimension of x can be arbitrary, because these inequalities are, in fact, one-dimensional.

Inequality (23) and inequalities (20) and (19) corresponding to it are the well-known Hardy inequality [7, Sec. 9.8]; therefore, the inequality $J \ge 0$ can be treated as its two-dimensional generalization if (12) and (16) are fulfilled.

(e) Thus, functionals (1), (3), (5), and (7) have a meaning not only for compactly supported x, u (which was initially assumed for simplicity) but for any pairs x, u from the space L_2 with the corresponding weight. We have the following simple proposition.

Proposition 1. *In functional* (3), *the pair (x, w) belongs to $L_2\,[0,\infty)$ with the weight $e^{-\tau}$ $\iff$ in functional* (5), *the pair (z, v) belongs to the "ordinary" $L_2\,[0,\infty)$ $\iff$ in functional* (7), *the pair (x, u) belongs to $L_2\,[0,\infty)$ $\iff$ in functional* (1), *the pair $(x,\, tu)$ belongs to $L_2\,[0,1]$.*

The nonnegativity of the functional J on the set of compactly supported pairs is equivalent to its nonnegativity on the set of pairs from the corresponding space L_2. (As a compactly supported pair for the space $L_2[0,1]$, we take a pair for which $x = u = 0$ on $[0,\varepsilon]$ for a certain $\varepsilon > 0$.)

Estmates (19), (20), and (23) for functionals (1), (3), and (5) imply that the belonging of the control to the corresponding space L_2 ensures the belonging of the phase component to same space L_2 as well. This is not the case for functional (7): the corresponding case $b = 0$ and $Q = 0$ leads, as was already noted, to the trivial inequality $\int u^2\,dt \geq 0$, which does not relate the phase component to the control.

Let us show how the representations in the form of "sum of squares" for functionals (1), (3), (5), and (7) are related to each other. If for functional (1) we have obtained

$$J = \int_0^1 \big((tu - Rx)^2 + (Mx, x)\big)\,dt, \qquad \frac{dx}{dt} = u,$$

then for (3), after the substitution $dt = e^{-\tau} d\tau$, $tu = -w$, we obtain

$$J = \int_0^\infty e^{-\tau}\,\big[(w + Rx)^2 + (Mx, x)\big]\,d\tau, \qquad \frac{dx}{d\tau} = w;$$

for functional (5) we have $e^{-\tau/2}x = z$, $e^{-\tau/2}w = v$, and

$$J = \int_0^\infty \big[(v + Rz)^2 + (Mz, z)\big]\,d\tau, \qquad \frac{dz}{d\tau} = -\frac{1}{2}\,z + v,$$

and then for (7) we have $u = -\frac{1}{2}\,z + v$, i.e., $v = \frac{1}{2}\,z + u$,

$$J = \int_0^\infty \left[\left(u + \left(R + \frac{1}{2}\right)z\right)^2 + (Mz, z)\right] d\tau, \qquad \frac{dz}{d\tau} = u.$$

5. We call attention to the following interesting property.

Lemma 5. *Let inequality* (12) *or inequality* (15) *equivalent to it hold for functional* (7) *with the equality sign. Then the functional* J *is positive, i.e.,* $J(x) > 0 \quad \forall\, x \neq 0$.

Indeed, if (12) is an equality, then equalities are also fulfilled in formula (11), and the matrix M vanishes. Therefore, the functional J reduces to the first term in (8), and if $J(x) = 0$, then $u + (S + bP)x = 0$, i.e., $\dot{x} = -(S + bP)x$. Since $x(0) = 0$, we obtain $x \equiv u \equiv 0$; this is what was required to be proved.

Thus, if (12) is fulfilled in the form of an equality, then, on the one hand, we always have $J(x) > 0$; on the other hand, one cannot decrease certain q_i or increase $|b|$, since inequality (12) is violated in this case, and by the same Theorem 1, we obtain $J(x) < 0$ for a certain x.

At first glance, this circumstance contradicts the apparent Legendre property of J (especially in formula (5): we have the identity matrix of u^2, i.e., the strengthened Legendre condition is fulfilled uniformly in t !) because it is known that if the Legendre functional is positive, then it is positive-definite, and, therefore, any functional that is close to it is also positive-definite. However, there is no contradiction here because our J is not Legendre: for functionals on $[0, \infty)$, the positivity of the coefficient of u^2 is not sufficient to be Legendre because the remaining terms are not weakly continuous. (This corresponds to the property that the operator $u \mapsto x$ mentioned in Lemma 4 is not completely continuous when it is considered on an infinite interval.)

Lemma 5 implies that the constant 4 in inequalities (19), (20), and (23) is sharp but is not attained. That is, on the one hand, it cannot decrease, and, on the other hand, for any nonzero function $x(t)$, these inequalities are strict, i.e., they are fulfilled with a certain $C(x) < 4$. This is a rather interesting peculiarity of inequalities for functions and their derivatives.

One more corollary of Lemma 5 is the possibility of constructing an example of optimal control problem on $[0, \infty)$ which has no solution.

Example 1. Consider the problem $J(x, u) \to \min$, where J is given, e.g., by formulas (6) and (7) with coefficients satisfying (12) in the form of the equality (for example, $b = q_1 = q_2 = 0$) under the constraint

$$\int_0^\infty |x|^2 \, dt = 1. \tag{24}$$

Let us show that the minimum in this problem is not attained. For this purpose, it is sufficient to show that $\inf J = 0$ (and then, by Lemma 5, it is certainly not attained).

Indeed, if for a given constraint, we have $\inf J = a > 0$, then, by virtue of the homogeneity, for any $x \in L_2\,[0, \infty)$, we have $J(x) \geq a \int |x|^2 \, dt$, i.e.,

$$\tilde{J}(x) = J(x) - a \int |x|^2 \, dt \geq 0. \tag{25}$$

But the functional $\tilde{J}$ has matrix $\tilde{Q} = Q - aE$, that violates the inequality (12) (because it is fulfilled as an equality for Q); therefore, by Theorem 1, (25) cannot hold for all x. We arrive at a contradiction.

We call attention to the fact that the proposed problem satisfies the standard requirement of convexity in u. Certainly, one can raise the objection that it does not satisfy the requirement of compactness in u. Then we consider the following example.

Example 1'. The same problem with the additional constraint $|u| \le 1$. Let us show that here $\inf J = 0$ still holds.

Take $b \ge 0$ for certainty. If the equality holds in (12), then there exists a unique c for which (11) is fulfilled; moreover it becomes an equality. In this case, $\sqrt{q_1} = b + c$, $\sqrt{q_2} = b - c$, the matrix $M = 0$, and the functional has the form

$$J = \int_0^\infty (u + Rx)^2 \, dt, \tag{26}$$

where

$$R = \begin{pmatrix} 0 & -(b-c) \\ (b+c) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{q_1} \\ \sqrt{q_2} & 0 \end{pmatrix}. \tag{27}$$

Since the matrix R has the characteristic equation $\lambda^2 + \sqrt{q_1 q_2} = 0$, it always has an eigenvalue $\lambda = i\omega$ with zero real part, and, therefore, Eq. (14), $\dot{x} = -Rx$, always has a solution of the form $x = \xi e^{i\omega t}$ for a certain real ω and a complex $\xi \ne 0$.

As in Lemma 3, for each N, we take a function $x_N(t)$ that coincides with this solution on the closed interval $[1, N]$, vanishes for $t = 0$ and $t > N + 1$, and is linear on the intervals $[0, 1]$ and $[N, N+1]$. Then, $|x_N(t)| \le \text{const}$, and for $u_N = \dot{x}_N$ the inequality $|u_N(t)| \le \text{const}$ also holds. Therefore,

$$J(x_N) = \int_0^1 (u + Rx)^2 \, dt \; + \int_N^{N+1} (u + Rx)^2 \, dt \le \text{const}$$

and

$$\int_0^\infty |x_N|^2 \, dt = \beta_N^2 \to \infty.$$

(Here, β_N^2 is a quantity of order N.) Then $\hat{x}_N = x_N / \beta_N$ satisfies constraint (24) and $|\hat{u}_N| \le \dfrac{\text{const}}{\beta_N} \to 0$ for it; therefore, for large N, the pair $(\hat{x}_N, \hat{u}_N)$ satisfies all the constraints and

$$J(\hat{x}_N) \le \frac{\text{const}}{\beta_N^2} \to 0 \, .$$

This implies $\inf J = 0$, which is what was required to be proved.

In the particular case where $b = q_1 = q_2 = 0$ (actually, this is a one-dimensional case), we obtain that in the problem

$$J = \int_0^\infty u^2 \, dt \to \min, \qquad \dot{x} = u, \quad x(0) = 0,$$

$$\int_0^\infty x^2 \, dt = 1,$$

and also in this problem with the additional constraint $|u| \le 1$, the minimum is not attained.

Similar considerations show that there is no solution in the following example of the "economics" type.

Example 2. Consider the problem

$$J = \int_0^\infty e^{-\tau}[w^2 + 2b(Px, w) + (Dx, x)]\, d\tau \to \min,$$

$$\frac{dx}{d\tau} = w, \qquad x(0) = 0,$$

$$\int_0^\infty e^{-\tau}(x, x)\, d\tau = 1, \tag{28}$$

where b and D satisfy condition (16) in the form of an equality (e.g., $b = 0$ and $d_1 = d_2 = -1/4$).

Here, as before, $\inf J = 0$, but it is not attained. Indeed, in this case, the corresponding functional (7), as we know, can be represented in the form (26) with the matrix R of the form (27) having an eigenvalue $\lambda = i\omega$, and then, according to Sec. 4, our functional can be represented in the form

$$J = \int_0^\infty e^{-\tau}\, (w + R'x)^2\, d\tau,$$

where $R' = R - \frac{1}{2}\, E$ has the eigenvalue $\lambda = -\frac{1}{2} + i\omega$. Then the equation $\dot{x} = -R'x$ has a solution $x(\tau) = \xi\, e^{(\frac{1}{2} + i\omega)\tau}$, and for the corresponding $x_N(\tau)$ (which coincides with $x(\tau)$ on the closed interval $[1, N]$ and is equal to zero outside $(0, N+1)$), we still have

$$\int_0^\infty e^{-\tau}|x_N|^2\, d\tau \;=\; \beta_n \to \infty.$$

Moreover, for $u_N(\tau) = \dot{x}_N(\tau)$, the estimate $|u_N(\tau)| \le \text{const} \cdot e^{\tau/2}$ is always satisfied so that still $J(x_N) \le \text{const}$. Then, as before, passing to $\hat{x}_N = x_N/\beta_N$, we obtain $J(\hat{x}_N) \to 0$; this is what was required to be proved.

However, note that when we pass to Problem $2'$ by adding the constraint $|u| \le c$, there is already no analogy with problem $1'$. The point is that in Example 2 the "vanishing" control $u = \xi\, e^{(\frac{1}{2} + i\omega)\tau}(\frac{1}{2} + i\omega)$ has the norm $\|u\|_\infty$ of order $e^{\frac{1}{2}N}$ (it is attained for $\tau = N$), while the left-hand side of (28) is a quantity of order N; therefore, for the normalized sequence $\hat{x}_N = x_N/\sqrt{N}$ satisfying (28), we have $\|\hat{u}_N\|_\infty \simeq e^{N/2}/\sqrt{N} \to \infty$, which means the violation of the constraint $\|u\|_\infty \le c$ for any c.

6. Relation with the theory of conjugate points. Let us now study the problem of the nonnegativity of functional (6), (7) using the classical Jacobi condition for conjugate points. The nonnegativity of J on the set of all compactly supported $x,\ u$ (which is equivalent to its nonnegativity on all $x,\ u \in L_2\,[0,\infty)$) is obviously equivalent to the fact that $\forall\, T > 0$ $J(x,u) \geq 0$ for all $x,\ u$ concentrated on the closed interval $[0,T]$.

Thus, we need that for any fixed $T > 0$ our functional satisfies the inequality

$$J_T = \int\limits_0^T [u^2 + 2b(Px,u) + (Qx,x)]\,dt \geq 0 \tag{29}$$

on all $x,\ u$ such that $u \in L_2\,[0,T]$, and

$$\dot{x} = u, \qquad x(0) = 0, \quad x(T) = 0. \tag{30}$$

We have a standard quadratic functional of the classical calculus of variations satisfying the strengthened Legendre condition. The Jacobi condition for it states that $J_T \geq 0$ if and only if the interval $(0,T)$ does not contain a point conjugate to $t=0$. Since T is arbitrary, there must be no points conjugate to $t=0$ on the whole semi-axis $(0,\infty)$. We want to find out for which b and Q this case is realized.

To find a conjugate point, let us write the Euler–Jacobi equation

$$\frac{d}{dt}\,(L_u) = L_x\,,$$

where L is the integrand in (29). Thus, we have the equation (of second order and with constant coefficients)

$$\ddot{x} = -(2b)P\dot{x} + Qx, \qquad x(0) = 0, \tag{31}$$

and the conjugate point T_* is determined by the equation $x(T_*) = 0$. (To be more precise, we are interested in the "first" conjugate point, that is the smallest $T_* > 0$ for which there exists a nontrivial solution of Eq. (31) vanishing at this point.)

The nonnegativity of (29) for any T is equivalent to the property that (31) vanishes nowhere on $(0,\infty)$. Therefore, Theorem 1 is equivalent to the following (a priori not obvious) assertion on the qualitative behavior of solutions of Eq. (31).

Theorem 2. *Any nontrivial solution of Eq.* (31) *does not vanish on the semi-axis* $(0,\infty)$ *iff the eigenvalues of the matrix* Q *are nonnegative and satisfy inequality* (12).

Proof. In the case where $Q = qE$ is a scalar matrix, this theorem can be easily proved directly. In this case, inequality (12) becomes

$$\sqrt{q} \geq b, \qquad \text{i.e.,} \quad q \geq b^2. \tag{32}$$

Let us consider the vector $x \in \mathbf{R}^2$ as a representation of a complex number $z \in \mathbf{C}$. Then the matrix P corresponds to the multiplication by i, and Eq. (31) becomes

$$\ddot{z} = -(2bi)\dot{z} + qz, \qquad z(0) = 0. \tag{33}$$

The corresponding characteristic equation $\lambda^2 + (2bi)\lambda - q = 0$ has the roots

$$\lambda = -bi \pm \sqrt{-b^2 + q}.$$

Under the root sign, we have exactly the quantity from inequality (32). Consider all possible cases.

(a) $q > b^2$, i.e., inequality (32) is strict. Then $\lambda = \pm a - bi$, where $a = \sqrt{q - b^2} > 0$, and

$$z = e^{-ibt}(c_1 \sinh at + c_2 \cosh at).$$

By virtue of the initial condition $z(0) = 0$, we have $c_2 = 0$; therefore, $c_1 \neq 0$ (we consider only nontrivial solutions!), and then $z(t) \neq 0$ for all $t > 0$. Thus, in this case, there is no conjugate point.

(b) $q = b^2$, i.e., (32) is fulfilled with the equality sign. Then $\lambda = -bi$ is the eigenvalue of multiplicity 2, and the general solution of Eq. (33) has the form

$$z = e^{-ibt}(c_1 t + c_2).$$

Taking the initial condition into account, we have $c_2 = 0$, $c_1 \neq 0$, and then $z(t) \neq 0$ for all $t > 0$ once again, i.e., the conjugate point is still absent.

(c) $0 < q < b^2$, i.e., (32) is not fulfilled. Then $\lambda = -ib \pm ia$, where $a = \sqrt{b^2 - q} < b$, i.e., $\lambda_1 = -i\mu_1$, and $\lambda_2 = -i\mu_2$, where $0 < \mu_1 < \mu_2$. Here,

$$z = c_1 e^{-i\mu_1 t} + c_2 e^{-i\mu_2 t}$$

and $c_1 + c_2 = 0$, i.e., $z = c(e^{-i\mu_1 t} - e^{-i\mu_2 t})$, $c \neq 0$. The relation $z(T_*) = 0$ means that

$$e^{i(\mu_2 - \mu_1)T_*} = 1.$$

The smallest solution of this equation is $T_* = 2\pi/(\mu_2 - \mu_1)$. Thus, in this case, the conjugate point does exist.

(d) $0 = q < b^2$; (32) is not fulfilled. Then $\lambda_1 = 0$, $\lambda_2 = -2bi$, and $z = c(-1 + e^{-2bi\,t})$, $c \neq 0$. We find the conjugate point $T_* = \pi/b$ from the relation $z(T_*) = 0$.

(e) $q < 0 < b^2$; (32) is not fulfilled. Then $\lambda_1 = i\mu_1$ and $\lambda_2 = -i\mu_2$, where $0 < \mu_1 < \mu_2$. Similarly to case (c), we have $z = c(e^{i\mu_1 t} - e^{-i\mu_2 t})$, $c \neq 0$, and the conjugate point is $T_* = 2\pi/(\mu_2 + \mu_1)$.

(f) Finally, consider the trivial case where $b = 0$. In this case, we have $\lambda^2 = q$ (and the functional actually becomes one-dimensional). If $q = 0$, i.e., if (32) is satisfied, then, taking the initial condition into account, we have $z = ct$, $c \neq 0$, and then $z(t) > 0$ for all $t > 0$.

If $q > 0$, i.e., if (32) is also fulfilled; then $z = c \sinh \omega t$, $\omega = \sqrt{|q|}$, $c \neq 0$, and then $z(t) > 0$ for all $t > 0$.

On the other hand, if $q < 0$, (32) is not satisfied, then $z = c \sin \omega t$, $\omega = \sqrt{|q|}$, and $c \neq 0$, and the condition $z(T_*) = 0$ yields the conjugate point $T_* = \pi/\omega$.

Thus, we have considered all possible cases and established that in each of them, (32) correctly points to the presence or absence of a conjugate point. Theorem 2 and Theorem 1 equivalent to it are proved. □

In the case of distinct eigenvalues $q_1 \neq q_2$, this method for proving Theorem 2 would require much more complicated calculations.

7. Connection with the frequency criterion. Let us point to one more possible method for studying our functional, namely, to the use of the so-called "frequency criterion" (see, e.g., [2, 10]). In essence, it consists of the following simple procedure (strange as it seems, this procedure is not mentioned in these works). As before, we consider the functional J of the form (7) on solutions of Eq. (6). If $J(x) \geq 0$ on all compactly supported x, then for any fixed $T > 0$, we obviously have $J(x) \geq 0$ on all functions x concentrated on the closed interval $[0,T]$. Any such function x and its derivative u can be expanded into the Fourier series on this interval. Using the complex notation, we have

$$x = \operatorname{Re}\left(\xi_0 + \sum_{k=1}^{\infty} \xi_k\, e^{i\omega_k t}\right), \qquad u = \operatorname{Re}\left(\sum_{k=1}^{\infty} i\omega_k\, \xi_k\, e^{i\omega_k t}\right), \tag{34}$$

where ξ_k are arbitrary vectors from $\mathbf{C}^2$, the series of whose squares converges, and $\omega_k = \dfrac{T}{2\pi}k$. (Obviously, it sufficies to consider finite sums of the above form, since they are dense in $L_2[0,T]$.)

Substitute (34) in functional (7). Note that in doing so, the products of distinct harmonics yield a function of the form $\xi' e^{i\omega' t}$, where $\omega' = \omega_k - \omega_l \neq 0$, whose integral over its period is equal to zero. Thus, J is decomposed into a sum of functionals for each harmonic taken separately. These harmonics can be conveniently calculated by the following formula: if arbitrary $\xi, \eta \in \mathbf{C}^2$ and real $\omega = \dfrac{T}{2\pi}k$ (k is an integer) are given, then the complex fuctions

$$z = \xi e^{i\omega t}, \qquad y = \eta e^{i\omega t},$$

satisfy the following relation:

$$\int_0^T (\operatorname{Re} z, \operatorname{Re} y)\, dt = \frac{1}{2}\operatorname{Re}\int_0^T \langle z, y\rangle\, dt = \frac{T}{2}\operatorname{Re}\langle \xi, \eta\rangle, \tag{35}$$

where we denoted by parentheses the componentwise inner product of two-dimensional vectors and by angle brackets, the complex inner product. (This relation is established by a direct calculation.)

Since the coefficients of all harmonics are independent of each other, we obtain the following requirement for each harmonic with serial number $k \geq 1$:

$$J = \mathrm{Re} \int\limits_0^T (\omega_k^2 |\xi_k|^2 - 2bi\omega_k (P\xi_k, \bar{\xi}_k) + (Q\xi_k, \bar{\xi}_k))\, dt \geq 0. \tag{36}$$

Let us study this inequality (omitting the subscript k). Let $\xi = f+ih,$ where $f,\ h \in \mathbf{R}^2.$ Then the integrand in (36) can be written in the form

$$\omega^2(|f|^2 + |h|^2) - 2i\omega b(P(f+ih),(f-ih)) + (Q(f+ih),(f-ih));$$

therefore, the real part of inequality (36) means that

$$\omega^2(f^2+h^2) - 4\omega b(Pf,h) + (Qf,f) + (Qh,h) \geq 0. \tag{37}$$

This inequality must be fulfilled for any vectors $f,\ h \in \mathbf{R}^2$ and for any ω that is a multiple of $T/2\pi$.

Now, note that since T is arbitrary, inequality (37) must be fulfilled for all $\omega > 0$. Moreover, if we replace ω by $-\omega$ and f by $-f$ in this inequality, then it does not change, and hence, it must be fulfilled for all $\omega \in \mathbf{R}.$ The consideration of the harmonic $k=0$ leads to the obvious requirement that $Q \geq 0,$ which is already contained in (37) (one should take $\omega = 0,\ h = 0$).

Thus, we arrive at the following proposition.

Proposition 2. *The functional J is nonnegative on all compactly supported functions iff* (37) *holds for any $f,\ h \in \mathbf{R}^2$ and any $\omega \in \mathbf{R}$.*

Proof. (a) Sufficiency. Let (37) be fulfilled. Then for any $T > 0$ and any harmonic on the closed interval $[0,T]$, (36) is fulfilled. Therefore, for each finite sum of such harmonics, we have $J \geq 0,$ and then by the mentioned density of finite sums in $L_2[0,T]$, we have $J \geq 0$ also for any pair $(x,u) \in L_2[0,T]$ (in particular, for such pairs with $x(T) = 0$). Since T is arbitrary, this implies that $J \geq 0$ for all compactly supported $(x,u) \in L_2\,[0,\infty).$

(b) We prove necessity by assuming the contrary. Let (37) be not fulfilled for certain $\omega,\ f,$ and $h.$ Then we obtain a violation of (36) for the corresponding harmonic (x,u) on its period $[0,T]$, i.e., $J(x,u) = -\alpha < 0.$ On its multiple period $[1, NT+1]$, we have $J(x,u) = -\alpha N$ for any N . Now, connecting $x(1)$ linearly with $x(0) = 0$ on the interval $[0,1],$ and $x(NT+1) = x(1)$ with $x(NT+2) = 0$ on the interval $[NT+1,\ NT+2],$ we obtain only a finite contribution to the integral (because $x(t)$ is bounded) so that for large N , we have a finitely supported pair $(\hat{x}, \hat{u})$ at which $J < 0$; this is what was required to be proved. □

The method of studying the integral quadratic functional on $[0,\infty)$ by passing to the expansion of $x,\ u$ into Fourier series was used in [3, 4, 6]; criterion (37) was also obtained in these works.

Let us analyze this criterion. Since we have a quadratic trinomial in ω, we take its discriminant to obtain that (37) is equivalent to the inequality

$$4b^2(Pf,h)^2 \le ((Qf,f)+(Qh,h))(f^2+h^2), \tag{38}$$

which must be fulfilled for any $f,\ h \in \mathbf{R}^2$. In this form, the criterion for the nonnegativity of J was obtained in [1]. Its deficiency is that (38) is an inequality of fourth degree with respect to an arbitrary pair $f,\ h,$ and it is not clear how to study it further.

Let us show that, in fact, (38) is equivalent to (12). As before, without loss of generality, we take $Q = \operatorname{diag}(q_1, q_2)$. Represent the vectors f and h in this basis in the coordinate form, i.e., take $f = (f_1, f_2)$ and $h = (h_1, h_2)$. Then (38) becomes

$$4b^2(f_1h_2 - f_2h_1)^2 \le (q_1(f_1^2+h_1^2) + q_2(f_2^2+h_2^2))(f_1^2+h_1^2+f_2^2+h_2^2). \tag{39}$$

If we introduce the vectors $x = (f_1, h_1)$ and $y = (f_2, y_2)$, then inequality (39) can be rewritten as

$$4b^2(Px,y)^2 \ \le\ (q_1x^2+q_2y^2)\,(x^2+y^2). \tag{40}$$

It is clear that for any fixed $|x|$ and $|y|$, the maximum of the left-hand side is attained for $x \perp y$ (i.e., the worst case is realized); then $|(Px,y)| = |x|\cdot|y|,$ and, therefore, (40) is equivalent to the inequality

$$4b^2x^2\,y^2 \ \le\ (q_1x^2+q_2y^2)\,(x^2+y^2) \tag{41}$$

for scalar quantities x^2 and y^2. For $|x| = 0$ or $|y| = 0$, it holds trivially, and, therefore, it is sufficient to verify it for $|x| > 0$ and $|y| > 0$. Then, setting $|y|^2 = \alpha|x|^2,$ we obtain that the following inequality must hold for all $\alpha > 0$:

$$4b\alpha \le (q_1 + q_2\alpha)(1+\alpha).$$

Dividing it by $\alpha,$ we obtain

$$4b \le (q_1+q_2) + \frac{q_1}{\alpha} + q_2\alpha. \tag{42}$$

As is known, the minimum of the right-hand side (the worst case once again) is attained for $q_1/\alpha = q_2\alpha,$ i.e., for $\alpha = \sqrt{q_1/q_2}.$ Then (42) transforms into the inequality

$$4b^2 \le (q_1+q_2) + 2\sqrt{q_1q_2}. \tag{43}$$

But the right-hand side here is a complete square $(\sqrt{q_1}+\sqrt{q_2})^2,$ and hence, (43) is equivalent to $2|b| \le \sqrt{q_1}+\sqrt{q_2},$ which is exactly the above inequality (12).

8. Justification of the method. Let us dwell on the justification of the method proposed in Sec. 3. Why does the addition of the expression $\frac{d}{dt}(Sx, x)$ under the integral sign lead to success? Note that for the problem of minimizing the functional (7) over the solutions of system (6), the function $\varphi(x) = (Sx, x)$ is the so-called Krotov function. Recall that in the general case, a Krotov function is a function $\varphi(x, t)$ for which the integrand, after the addition of $\frac{d}{dt}\varphi(x, t)$, attains the minimum value over all x, u, that are not related by Eq. (6), on the examined trajectory $x^0(t)$, $u^0(t)$. If such a (smooth) function exists, then it is easily established that this trajectory yields a strong minimum in the problem. (For more details, see [5, 9].) Thus, the existence of a Krotov function is a sufficient condition for a strong minimum. In studying functional (7), we actually take $x^0(t) \equiv u^0(t) \equiv 0$ as the examined trajectory, and the minimality of this trajectory means that $J \geq 0$ on subspace (6).

Since functional (7) is quadratic, it is natural to seek a Krotov function as a quadratic form, i.e., $\varphi(x) = (Sx, x)$. A nontrivial (and, in the general case, not yet sufficiently studied) question is whether the existence of a Krotov function is a necessary condition for optimality of the examined trajectory? In our case, this question is formulated as follows: *is the existence of the required quadratic form (Sx, x) necessary for the nonnegativity of the functional* (7)? In other words, let there be no symmetric matrix S for which the integrand in (8) is nonnegative for independent x, u (i.e., there does not exist S for which $M \geq 0$). Then, why does there exist x, u satisfying (6) for which $J < 0$? In Sec. 3, this property was established by presenting a particular pair (x, u) for the two-dimensional case. A.A. Milyutin demonstrated (see [11]) that it is also valid for the case of an arbitrary linear system $\dot{x} = Ax + Bu$ with constant coefficients under arbitrary dimensions of x and u. This property is of interest because it is a quite rare case where one can assert that the existence of a Krotov function is *necessary* for optimality. Here, we present its proof for the case of the simplest control system $\dot{x} = u$, $x \in \mathbf{R}^n$ (with arbitrary n), in which the proof is technically much easier than in the general case.

Let us consider the quadratic functional

$$J = \int_0^\infty (u^2 + (Vx, u) + (Qx, x))\, dt \tag{44}$$

on the subspace $\mathcal{L}$:

$$\dot{x} = u, \qquad x(0) = 0, \tag{45}$$

where x, $u \in \mathbf{R}^n$, Q is a symmetric matrix, and V is a skew-symmetric matrix (the symmetric part of V yields zero in the integral). We are interested in the following problem: when is $J \geq 0$ on the subspace $\mathcal{L}$?

As before, we take an arbitrary symmetric matrix S and, adding $\frac{d}{dt}(Sx, x)$ under the integral sign, we obtain

$$J = \int_0^\infty \big([u + (S+V)x]^2 + (M(S)x, x)\big)\, dt, \tag{46}$$

where

$$(M(S)x, x) = (Qx, x) - (Sx + Vx)^2, \tag{47}$$

and $M(S)$ is a symmetric matrix.

Following Milyutin, we denote by $\lambda(S)$ the minimum eigenvalue of the matrix $M(S)$, i.e.,

$$\lambda(S) = \min_{|x|=1} (M(S)x, x),$$

and set $\lambda^0 = \sup \lambda(S)$ over all symmetric S. If $\lambda^0 > 0$, then $M(S) > 0$ for a certain S, and, therefore, $J \geq 0$. If $\lambda^0 = 0$, then for all $\varepsilon > 0$, we consider the functional $J_\varepsilon = J + \int \varepsilon(x, x)\, dt$, for which $Q_\varepsilon = Q + \varepsilon E$, and therefore, $\lambda^0_\varepsilon = \lambda^0 + \varepsilon > 0$. Hence, $J_\varepsilon \geq 0$, and then, passing to the limit, we also obtain that $J \geq 0$ on $\mathcal{L}$.

It remains to consider the case where $\lambda^0 < 0$. Our purpose is to show that, in this case, there exists $x \in \mathcal{L}$ for which $J(x) < 0$. Thus, we should prove the following theorem.

Theorem 3 (A.A. Milyutin)**.** *The functional* $J \geq 0$ *on* $\mathcal{L}$ *iff* $\lambda^0 \geq 0$.

This theorem justifies the procedure of searching for an appropriate matrix S proposed above. The proof uses the following property of matrices established by Milyutin.

Let R be an arbitrary $(n \times n)$-matrix.

Lemma 6 ([11])**.** *The following two conditions are equivalent*:

(a) *for any symmetric matrix* S, *we have*

$$\max_{|x|=1} (Rx, Sx) \geq 0\,; \tag{48}$$

(b) *the matrix* R *has an eigenvalue with zero real part.*

Proof. Show that $(b) \Longrightarrow (a)$. If $\lambda = 0$ is an eigenvalue of the matrix R, then there exists a vector x_0 such that $|x_0| = 1$ and $Rx_0 = 0$. Then for all S, we have $(Rx_0, Sx_0) = 0$, and, therefore, (48) is fulfilled.

Let $\lambda = i\omega \neq 0$ be an eigenvalue of the matrix R. Then, as is known, the equation $\dot{x} = Rx$ has a periodic solution of the form $x^0(t) = f \sin \omega t + h \cos \omega t$, where the vectors $f,\ h \in \mathbf{R}^n$ are linearly independent, and, therefore, $x^0(t) \neq 0$ everywhere. Take an arbitrary symmetric matrix S and consider the function $m(t) = (Sx^0(t), x^0(t))$. Since it is periodic, it has a maximum point t_*. At this point, we have $\frac{d}{dt} m(t_*) = 2(Sx^0(t_*), Rx^0(t_*)) = 0$, and since $x^0(t_*) \neq 0$, this implies (48), which was required to be proved.

Prove the implication $(a) \Longrightarrow (b)$ by assuming the contrary. Let all eigenvalues λ of the matrix R have $\operatorname{Re}\lambda \neq 0$. We need to show that there exists a symmetric matrix S for which (48) is violated. This is equivalent to the fact that there is a quadratic form $\varphi = (Sx, x)$ which has a negative derivative on any nonzero solution of the system $\dot{x} = Rx$, i.e., $\dfrac{d}{dt}\varphi(x(t)) = (Sx, Rx) < 0$, i.e., φ is a Lyapunov function for this system. But the existence of a quadratic Lyapunov function does not depend on the basis in which the system is considered, so we can seek this function in the basis where the matrix R reduces to the real Jordan form. Obviously, it is enough to consider the case of a single real Jordan block (since any block corresponds to an invariant subspace of the matrix R), i.e., we can assume that R is the block corresponding to a pair of complex-conjugate eigenvalues λ, $\bar{\lambda}$. In addition, it can be assumed that $\operatorname{Re}\lambda < 0$ (otherwise, we replace R by $-R$ and S by $-S$). As is known from the theory of ordinary differential equations, in this case the system $\dot{x} = Rx$ has a quadratic Lyapunov function $f(x) = (Sx, x)$. Now it remains to sum up the quadratic forms corresponding to all Jordan blocks and return to the initial basis. The lemma is proved. $\square$

We also need the following nontrivial fact. Denote by Σ the unit sphere in $\mathbf{R}^n$.

Lemma 7 ([11]). *Let a nonzero subspace $\Gamma_0 \subset \mathbf{R}^n$ and an $(n \times n)$-matrix R be such that, for any symmetric $(n \times n)$-matrix S, we have*

$$\max_{x \in \Gamma_0 \cap \Sigma} (Rx, Sx) \geq 0. \tag{49}$$

Then there exists a nonzero subspace $\Gamma \subset \Gamma_0$ invariant with respect to R (i.e., $R\Gamma \subset \Gamma$) for which (49) *is also fulfilled.*

Proof. Let Γ_1 be the orthogonal complement to Γ_0, i.e., let $\mathbf{R}^n = \Gamma_0 \oplus \Gamma_1$. Denote by π_0 and π_1 the orthogonal projections on Γ_0 and Γ_1, respectively. For any operator $A : \mathbf{R}^n \to \mathbf{R}^n$, we consider the operators $A_0 = \pi_0 A$ and $A_1 = \pi_1 A$, so that $A = A_0 + A_1$. Then, the following expansion always holds:

$$(Rx, Sx) = (R_0x, S_0x) + (R_1x, S_1x). \tag{50}$$

Note that on the subspace Γ_0, the operator A coincides with a certain symmetric operator S if and only if A_0 is a symmetric operator on Γ_0, while A_1 can be completely arbitrary. (It is convenient to imagine Γ_0 and Γ_1 as coordinate subspaces.)

Define the subspace $L = \{x \in \Gamma_0 \,|\, Rx \in \Gamma_0)\}$. If $L = \Gamma_0$, i.e., if Γ_0 is invariant for R, then everything is proved, and, therefore, it is necessary to consider the case $L \neq \Gamma_0$.

Let us prove that always $L \neq \{0\}$. In other words, the operator R_1 always has a nontrivial kernel on Γ_0. If this is not so, i.e., if $R_1x \neq 0$ on $\Gamma_0 \cap \Sigma$, then we take a symmetric matrix S such that $S_0x = 0$ and $S_1x = -R_1x$ on Γ_0. On $\Gamma_0 \cap \Sigma$, we obtain for this matrix that

$$(Rx, Sx) = -(R_1x, S_1x) = -|R_1x|^2 < 0,$$

which contradicts the condition of the lemma.

Thus, L is a nonzero subspace in Γ_0, and we assume that $L \neq \Gamma_0$. We assert that (49) is fulfilled for L as well, i.e., for any symmetric matrix S, we have

$$\max_{x \in L \cap \Sigma} (Rx, Sx) \geq 0. \tag{51}$$

Assume the contrary. Then there exist a symmetric matrix S and a number $\alpha_0 > 0$ such that

$$(Rx, Sx) \leq -\alpha_0 < 0 \qquad \text{on } L \cap \Sigma. \tag{52}$$

We decompose Γ_0 into the orthogonal sum $\Gamma_0 = L \oplus H$, and for $x \in \Gamma_0$ we write $x = x_L + x_H$. By definition, $RL \subset \Gamma_0$ and $RH \cap \Gamma_0 = \{0\}$.

Now, take a symmetric matrix $\hat{S}$ such that

$$\hat{S}_0 = S_0, \qquad \hat{S}_1 = -NR_1 \qquad \text{on } \Gamma_0.$$

According to (50), the following relation is fulfilled on Γ_0 for this matrix:

$$(Rx, \hat{S}x) = (R_0x, S_0x) - N|R_1x|^2. \tag{53}$$

Since $RL \subset \Gamma_0$, we have $R_1L = 0$, and, therefore, for each $x = x_L + x_H$, we obtain $R_1x = R_1x_H$; since $RH \cap \Gamma_0 = \{0\}$, we have $|R_1x_H|^2 \geq \alpha_1|x_H|^2$ for a certain $\alpha_1 > 0$.

For $x = x_L + x_H \in \Gamma_0$, let us calculate the first term on the right-hand side (53), i.e.,

$$(R_0x, S_0x) = (R_0x_L, S_0x_L) + (R_0x_L, S_0x_H) + (R_0x_H, S_0x_L) + (R_0x_H, S_0x_H).$$

Here, according to (52), $(R_0x_L, S_0x_L) \leq -\alpha_0|x_L|^2$, and therefore, the left-hand side of (53) satisfies the following estimate on Γ_0 with certain constants β and γ:

$$(Rx, \hat{S}x) \leq -\alpha_0\,|x_L|^2 + 2\beta|x_L| \cdot |x_H| + \gamma|x_H|^2 - N\alpha_1|x_H|^2.$$

Here, the right-hand side is a two-dimensional quadratic form in $|x_L|$ and $|x_H|$. Obviously, it is negative-definite for a sufficiently large N, which implies the violation of (51) for $\hat{S}$.

Thus, we have shown that if the subspace Γ_0 is not invariant with respect to R, then there exists a nonzero subspace L of a smaller dimension in it for which (49) is still fulfilled, i.e., we have (51). Continuing this process, in a finite number of steps we arrive at an invariant subspace L. The lemma is proved. □

Let us now turn to the determination of λ^0.

Lemma 8. *The* $\sup \lambda(S)$ *over all symmetric* S *is attained, i.e., there exists* S *for which* $\lambda^0 = \lambda(S)$.

Proof. It is clear in advance that $\lambda^0 > -\infty$. Let a sequence S_k be such that $\lambda(S_k) \to \lambda^0$. If the sequence of norms $\|S_k\|$ is bounded, then one can assume that $S_k \to S_0$, and then, the limit relation obviously holds: $\lambda(S_0) = \lambda^0$ (since the minimum of the quadratic form on the unit sphere continuously depends on its coefficients).

Assume now that $\|S_k\| \to \infty$ (on a certain subsequence). In this case, there exist vectors x_k such that $|x_k| = 1$ and $|S_k x_k| \to \infty$. Then, according to (47), the leading term in $(M(S_k)x_k, x_k)$ is $-|S_k x_K|^2 \to -\infty$, and, therefore,

$$\lambda(S_k) \;=\; \min_{|x|=1} (M(S_k)x, x) \;\le\; (M(S_k)x_k, x_k) \to \; -\infty,$$

which contradicts the relation $\lambda(S_k) \to \lambda^0$. The lemma is proved. $\square$

Now we can give a proof of Theorem 3.

Proof of Theorem 3. As was already mentioned, it suffices to consider the case $\lambda^0 < 0$.

For any symmetric matrix S , there exists a nonzero subspace $\Gamma(S) \subset \mathbf{R}^n$ (which is invariant for the matrix S) such that

$$\operatorname{Arg} \min_{x \in \Sigma} (M(S)x, x) \;=\; \Gamma(S) \cap \Sigma.$$

(Indeed, the quadratic form $(M(S)x, x) - \lambda(S)(x, x) \ge 0$ on $\mathbf{R}^n$ and has a nontrivial subspace of zeros; this is exactly the desired $\Gamma(S)$.) Now, take any S_0 at which $\max \lambda(S)$ is attained. Therefore, it is a solution of the problem

$$\lambda(S) = \min_{x \in \Sigma} (M(S)x, x) \to \max_S,$$

where x plays the role of a parameter.

Denote $\Gamma_0 = \Gamma(S_0)$ and introduce the matrix $R = S_0 + V$ so that

$$M(S_0) = (Qx, x) - (Rx)^2. \tag{54}$$

A key point in the proof of Theorem 3 is the following fact.

Proposition 1. *For any symmetric matrix $\bar{S}$, we have*

$$\max_{x \in \Gamma_0 \cap \Sigma} (Rx, \bar{S}x) \ge 0. \tag{55}$$

Proof. This follows from the formula of the directional derivative of the minimum function. Consider the symmetric matrix $S_\varepsilon = S_0 + \varepsilon \bar{S}$ for small $\varepsilon > 0$. With this matrix, we associate

$$\lambda(S_\varepsilon) = \min_{x \in \Sigma} (M(S_\varepsilon)x, x),$$

which is a function of ε. Since S_0 is a maximum point of $\lambda(S)$, we always have $\lambda(S_\varepsilon) \le \lambda(S_0)$; therefore, the right derivative (as $\varepsilon \to 0+$) $\dfrac{d}{d\varepsilon}\lambda(S_\varepsilon) \le 0$. Let us calculate this derivative.

Recall that if $\varphi(\varepsilon) = \min_{x \in K} \Phi(\varepsilon, x)$, where K is a compact set and Φ is a smooth function, then

$$\varphi'(0+) = \min_{x \in K_0} \Phi'_\varepsilon(0, x),$$

where $K_0 = \operatorname{Arg} \min \Phi(0, x) \mid x \in K$.

In our case, $\Phi(\varepsilon, x) = (M(S_\varepsilon)x, x)$, $\operatorname{Arg\,min}\{\Phi(0,x) | x \in \Sigma\} = \Gamma_0 \cap \Sigma$, and, according to (54), $\Phi'_\varepsilon(0, x) = -2(Rx, \bar{S}x)$. Therefore,

$$\frac{d}{d\varepsilon}\lambda(S_\varepsilon)\bigg|_{\varepsilon=0+} = \varphi'(0+) = \min_{x\in\Gamma_0\cap\Sigma}(-2Rx, \bar{S}x) \le 0,$$

which is equivalent to (55). □

Thus, (55) is established. By Lemma 7, there exists a nonzero subspace $\Gamma \subset \Gamma_0$ such that for any symmetric matrix $\bar{S}$,

$$\max_{x\in\Gamma\cap\Sigma}(Rx, \bar{S}x) \ge 0, \tag{56}$$

and, moreover, $R\Gamma \subset \Gamma$. This implies that one can pass to the restriction of R to the subspace Γ, i.e., to the operator $R : \Gamma \to \Gamma$, and the inequality (56) holds for any $\bar{S}$ that is symmetric on this subspace. Then, by Lemma 6, the operator R restricted on Γ has an eigenvalue $\lambda = i\omega$ with zero real part.

The latter means that in the subspace Γ, there exists a nonzero solution of the equation $\dot{x} = -Rx$, and the first square in (46) at this solution is equal to zero by definition. (Recall that $R = S_0 + V$.) If $\omega = 0$, then this solution is $x(t) \equiv x_0 \in \Gamma$; if $\omega \ne 0$, then this solution is $x(t) = f \sin\omega t + h\cos\omega t$, where $f,\ h \in \Gamma$. Since $(Mx, x) = \lambda^0 < 0$ on the subspace Γ_0 (the more so on Γ), at any of these solutions one can accumulate an arbitrarily large integral of the second term in (46). Now, repeating the arguments from the proof of Theorem 1, we obtain a compactly supported $\hat{x}(t)$ at which $J(\hat{x}) < 0$. Theorem 3 is proved. □

Conclusions. In this paper, we have considered the simplest nontrivial case of a quadratic functional with the degenerate Legendre condition by transforming it into a functional with "good" coefficients but on the semiaxis $[0, \infty)$. On the semiaxis (to be more precise, on the spaces of infinite measure), the integral functionals qualitatively differ in their properties from the integral functionals on closed intervals (i.e., on the spaces of finite measure); they still have a singularity. This is related to the fact that the integral operators on the spaces of infinite measure are not completely continuous in general.

We have succeeded in obtaining exact formulas for the nonnegativity of the examined functional only in the two-dimensional case. Even for the three-dimensional case, this question remains open.

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