A NONLOCAL LYUSTERNIK ESTIMATE AND ITS APPLICATION TO CONTROL SYSTEMS WITH SLIDING MODES

A.V. Dmitruk

Central Economics and Mathematics Institute,
Russian Academy of Sciences,
Moscow 117418, Nakhimovskii prospekt, 47.
e-mail: dmitruk@member.ams.org

Abstract

The property of so-called uniform covering is proved for a family of linear operators generated by a differential equation linear with respect to functional parameters. Using this fact, a uniform estimate of the distance to the level set of a nonlinear operator is obtained on a broader set than a neighborhood of the examined point in a Banach space. This allows to prove an approximation theorem for a system with sliding mode controls and endpoint state constraints.

Key words: nonlinear operator, covering, weak-* topology, sliding modes.

1. COVERING FOR A FAMILY OF LINEAR OPERATORS

Consider a Banach space $W = AC^{(m)} \times (L^r_\infty)^N$ on an interval [0,T] with elements $\bar x \in AC^{(m)}$ (absolute continuous $m-$ vector functions) and $\bar u = (\bar u^1, \ldots, \bar u^N)$ (each $\bar u^i \in L^r_\infty$). For any $\bar w = (\bar x, \bar u)$ we set $||\bar w|| = |\bar x(0)| + ||\dot{\bar x}||_1 + \sum ||\bar u^i||_\infty$. Denote $Z = L^m_1 \times \mathbb{R}^s$, and let for any $\alpha = (\alpha^1, \ldots, \alpha^M) \in L^M_\infty$ and any $\beta = (\beta^1, \ldots, \beta^N) \in L^N_\infty$ be given a linear operator $P[\alpha, \beta] : W \longrightarrow Z$, acting as follows: $(\bar x, \bar u) \mapsto (\bar\xi, \bar\kappa)$, where

$$\dot{\bar x} - \sum_1^M \alpha^i(t)\, A^i(t)\, \bar x - \sum_1^N \beta^j(t)\, B^j(t)\, \bar u^j = \bar\xi \in L^m_1,$$

$$K_0\,\bar{x}(0) + K_T\,\bar{x}(T) = \ \bar{\kappa} \in \mathbb{R}^s. \tag{1}$$

Here, A^i, B^j are measurable essentially bounded matrices, K_0 and K_T are constant $s \times m$ -matrices. We assume that the pair (α, β) belongs to a bounded set $S \subset L_\infty^{M+N}$. Fix a pair $(\alpha_0, \beta_0) \in S$, where $\alpha_0 = (\alpha_0^1, \ldots, \alpha_0^M)$ and $\beta_0 = (\beta_0^1, \ldots \beta_0^N)$.

Theorem 1. Let the pair (α_0, β_0) be such that the operator $P[\alpha_0, \beta_0]$ is onto. Then there exists a weak-* neighborhood $\mathcal{V}[\alpha_0, \beta_0]$ of this pair and a constant $c > 0$ such that, for any pair $(\alpha, \beta) \in \mathcal{V}[\alpha_0, \beta_0] \cap S$, the operator $P[\alpha, \beta]$ is $c-$ covering, i.e.,

$$P[\alpha, \beta]\,(D_1^W) \ \supset \ D_c^Z, \tag{2}$$

where D_ρ^W is the closed ball of radius ρ in the space W centered at the origin.

Proof. To simplify notation, we will write $\sum \alpha^i A^i \bar{x} = \alpha A\,\bar{x}$ and $\sum \beta^j B^j \bar{u}^j = \beta B\,\bar{u}$, where A and B are some 3-rank tensors, and $\bar{u} \in (L_\infty^r)^N$.

Since the space L_1 is separable and the set S is bounded, the weak-* topology on S is metrizable. Then, supposing that the assertion of the theorem is not true, there exists a sequence $(\alpha_n, \beta_n) \stackrel{\text{wk-*}}{\longrightarrow} (\alpha_0, \beta_0)$ and numbers $\delta_n \to 0+$ such that $\forall\, n$ the image set $P[\alpha_n, \beta_n]\,(D_1^W)$ does not contain the ball $D_{\delta_n/2}^Z$. (Here the subscript n indicates the number of the term in the sequence, not the component of the vector α nor β.) Since these image sets are convex, there exist linear functionals (ψ_n, μ_n) from the conjugate space Z^*, i.e., $\psi_n \in L_\infty^m = (L_1^m)^*$ and $\mu_n \in \mathbb{R}^s$, such that $||\psi_n|| + |\mu_n| = 1$, and

$$(\psi_n, \mu_n)\,(P[\alpha_n, \beta_n]\,(D_1^W)) \ \le \delta_n \ \to 0. \tag{3}$$

The last relation means that $\forall\,(\bar{x}, \bar{u}) \in W$ with $||\bar{x}||_{AC} + ||\bar{u}||_\infty \le 1$, the following inequality holds:

$$\int_0^T \psi_n(t)\,(\dot{\bar{x}} - \alpha_n(t)A(t)\,\bar{x} - \beta_n(t)B(t)\,\bar{u})\,dt +$$

$$+ \mu_n(K_0\,\bar{x}(0) + K_T\,\bar{x}(T)) \ \le \ \delta_n \to 0. \tag{4}$$

Let us analyze the obtained relation (4).
Set $\bar{x} = 0$. Then we have

$$\sup_{||\bar{u}||_\infty \le 1} \int_0^T (\psi_n \beta_n B(t)\,\bar{u})\,dt \ \le \delta_n \to 0,$$

whence obviously

$$||\psi_n \beta_n B||_1 \to 0. \tag{5}$$

Now, set $\bar u = 0$. Then

$$\sup_{||\bar x||_{AC}\le 1} \Big[\int_0^T \psi_n(t)\,(\dot{\bar x} - \alpha_n(t)A(t)\,\bar x)\,dt +$$
$$+ \mu_n(K_0\,\bar x(0) + K_T\,\bar x(T))\Big] \;\le\; \delta_n \to 0. \tag{6}$$

Take an $m-$ vector function $\varphi_n(t)$ satisfying the equation

$$\dot\varphi_n = -\psi_n\alpha_n A(t), \qquad \varphi_n(T) = -\mu_n K_T. \tag{7}$$

Then,

$$\int_0^T -(\psi_n\alpha_n A)\,\bar x\,dt = \int_0^T \dot\varphi_n \bar x\,dt =$$
$$= \varphi_n\bar x\,\big|_0^T - \int_0^T \varphi_n\dot{\bar x}\,dt,$$

hence (6) can be rewritten as

$$\sup_{||\bar x||_{AC}\le 1} \Big[\int_0^T (\psi_n - \varphi_n)\,\dot{\bar x}\,dt \;+\; (\mu_n K_0 - \varphi_n(0))\,\bar x(0)\Big] \;\le\; \delta_n \to 0. \tag{8}$$

But, here the variables $\dot{\bar x} \in L_1^m$ and $\bar x(0)\in \mathbb{R}^m$ can be chosen independently of each other, whence (setting each of them to zero in turn) we get

$$||\psi_n - \varphi_n||_\infty \;\to 0, \tag{9}$$

and

$$|\,\mu_n K_0 - \varphi_n(0)\,| \;\to 0. \tag{10}$$

From (9) we have $\psi_n = \varphi_n + \rho_n$, where $||\rho_n||_\infty \to 0$, and, in view of (7),

$$\dot\varphi_n = -\varphi_n\alpha_n A(t) + \sigma_n, \qquad ||\sigma_n||_\infty \to 0. \tag{11}$$

Without loss of generality, we assume $\mu_n \to \mu_0$ for some μ_0. Then, (7) implies that $\varphi_n(T) \to -\mu_0 K_T$. Since equation (11) is linear w.r.t. α_n, and $\alpha_n \overset{\text{wk-*}}{\longrightarrow} \alpha_0$, in this case, as is well known, φ_n uniformly converge to the solution of the equation

$$\dot\varphi_0 = -\varphi_0\alpha_0 A(t), \qquad \varphi_0(T) = -\mu_0 K_T, \tag{12}$$

i.e., $\varphi_n = \varphi_0 + \tilde\varphi_n$, where $||\tilde\varphi_n||_\infty \to 0$. Thus, $\psi_n = \varphi_0 + (\tilde\varphi_n + \rho_n)$, where $||\tilde\varphi_n + \rho_n||_\infty \to 0$, and $\mu_n \to \mu_o$. Therefore,

$$||\varphi_0||_\infty + |\mu_0| = 1. \tag{13}$$

Besides, we can replace in (6) ψ_n by φ_0 and μ_n by μ_0, and then replace α_n by its weak-* limit α_0. As a result we obtain that

$$\int_0^T \varphi_0\,(\dot{\bar x} - \alpha_0 A\,\bar x)\,dt \;+\; \mu_0(K_0\,\bar x(0) + K_T\,\bar x(T)) = 0 \tag{14}$$

for any $\bar{x} \in AC^{(m)}$.

Further, from (5) we get $||\varphi_0 \beta_n B||_1 \to 0$, hence $\int \varphi_0 \beta_n B\, \bar{u}\, dt \to 0$ for any $\bar{u} \in (L^r_\infty)^N$, and, since $\beta_n \xrightarrow{\text{wk-*}} \beta_0$, we have

$$\int_0^T \varphi_0 \beta_0 B\, \bar{u}\, dt = \int_0^T \sum_1^N \varphi_0 \beta_0^j B^j\, \bar{u}^j\, dt = 0 \tag{15}$$

(whence, dropping again $\bar{u}$, we get $\varphi_0 \beta_0 B = 0$).

Relations (14) and (15) together with normalization (13) mean that $\operatorname{Im} P[\alpha_0, \beta_0]$ lies in a proper subspace of Z defined by the equation $(\varphi_0, \bar{\xi}) + (\mu_0, \bar{\kappa}) = 0$, which contradicts the surjectivity of $P[\alpha_0, \beta_0]$. Theorem 1 is proved. □

A slight modification of the given proof allows one to obtain a similar theorem for the case when the space Z consists of three components: $Z = L_1^m \times \mathbb{R}^s \times (L^q_\infty)^N$, and the additional component of the operator $P[\alpha, \beta]$ acts as follows:

$$(\bar{x}, \bar{u}) \mapsto \{\, \Phi_j(t)\, \bar{x} + H_j(t)\, \bar{u}^j = \bar{\eta}^j \in L^q_\infty, \quad j = 1, \ldots, N\}, \tag{16}$$

where the matrices Φ_j and H_j of corresponding dimensions are measurable and essentially bounded, and all $H_j(t)$ have uniformly full rank, i.e., they have essentially bounded right inverse matrices $H_j^+(t): \; H_j(t)\, H_j^+(t) = I$ – the identity $q \times q-$ matrix. An equivalent requirement: $det\,(H_j(t)\, H_j^*(t)) \geq \text{const} > 0$. The corresponding version of Theorem 1 for this case we denote by Theorem $1'$.

2. COVERING FOR THE DERIVATIVE OF A SYSTEM WITH SLIDING MODES

Consider now the following control system (involving the so-called sliding modes):

$$\begin{gathered} \dot{x} - \sum_1^N \alpha^i(t) f(x, u^i, t) = 0, \\ K(x(0), x(T)) = 0, \\ g(x, u^i, t) = 0, \quad i = 1, \ldots, N, \\ \sum_1^N \alpha^i(t) - 1 = 0. \end{gathered} \tag{17}$$

Here $x \in AC^{(m)}[0, T]$, all $u^i \in L^r_\infty$, $\alpha^i \in L^1_\infty$, the function K is defined and smooth on $\mathbb{R}^{2m}$, the functions f, g with all their first order derivatives w.r.t. x, u are measurable in t, equicontinuous in (x, u) for all $t \in [0, T]$, and bounded for all bounded $x(t)$, $u(t)$. Denote for brevity the pair $(x(0), x(T)) = \rho$.

The left hand sides of equalities (17) define a nonlinear operator $F(x,u,\alpha)$ that acts from the space $W = AC^{(m)} \times (L^r_\infty)^N \times L^N_\infty$ into $Z = L^m_1 \times \mathbb{R}^s \times (L^q_\infty)^N \times L_\infty$, with the derivative $F'(x,u,\alpha) = P[x,u,\alpha] : W \longrightarrow Z$ acting as follows: $(\bar x, \bar u, \bar\alpha) \mapsto (\bar\xi, \bar\kappa, \bar\eta, \bar\nu)$, where

$$\begin{gathered}
\dot{\bar x} - \sum \alpha^i f'_x(x,u^i,t)\bar x - \sum \alpha^i f'_u(x,u^i,t)\bar u^i - \sum \bar\alpha^i f(x,u^i,t) \;=\; \bar\xi, \\
K'_{x(0)}(\rho)\,\bar x(0) + K'_{x(T)}(\rho)\,\bar x(T) \;= \bar\kappa, \\
g'_x(x,u^i,t)\bar x + g'_u(x,u^i,t)\bar u^i \;= \bar\eta^i, \quad i = 1,\dots,N, \\
\sum \bar\alpha^i(t) = \bar\nu.
\end{gathered} \tag{18}$$

Let us first establish the following property that might be of intrinsic interest.

Lemma 1. Let an operator $A : C^{(m)} \times L^r_\infty \to L^q_\infty$ acting by the rule

$$(\bar x, \bar u) \mapsto \; \Phi(t)\bar x + H(t)\bar u \;=\; \bar\eta \in L^q_\infty,$$

where Φ and H are measurable essentially bounded matrices, be onto. Then, the matrix $H(t)$ satisfies the above uniform full rank condition, i.e., has an essentially bounded right inverse.

Proof. Consider first the case $m = r = q = 1$, i.e., when

$$A(\bar x, \bar u) = \; \varphi(t)\bar x + h(t)\bar u \; \in L_\infty[0,T].$$

Since A is onto, we have

$$A\,(D^C_1 \times D^{L_\infty}_1) \;\supset\; D^{L_\infty}_a \tag{19}$$

for some $a > 0$. We have to prove that $vraimin\,|h(t)| > 0$. Suppose, on the contrary, that $vraimin\,|h(t)| = 0$. Then $|h(t)| \le a/3$ a.e. on a set E of positive measure. By the Lusin's $C-$ property, E contains a closed set M of positive measure on which the function $\varphi(t)$ is continuous. Let us restrict the spaces C and L_∞ to this set M. Obviously, inclusion (19) still holds for these restricted spaces.
Take any discontinuous function $\hat\eta \in L_\infty(M)$ with $||\hat\eta||_\infty \le a$ having the oscillation $> a$ at some point $\theta \in M$ (hence, θ is not isolated in M). The ball $D_{a/3}(\hat\eta)$ obviously contains only discontinuous functions (because their oscillations at θ are greater than $a/3$), therefore it has no common points with the set

$$Z = \; \{\, \bar z(t) = \varphi(t)\bar x(t) \mid \; \bar x \in C(M), \;\; ||\bar x||_C \le 1\},$$

because the last one consists of continuous functions. However, from (19) we have

$$\hat\eta = \bar z + h(t)\bar u \quad \text{for some } \; \bar z \in Z, \;\; ||\bar u||_\infty \le 1,$$

whence $||\bar{z} - \hat{\eta}||_\infty \le ||hu||_\infty \le a/3$, and thus $\bar{z} \in D_{a/3}(\hat{\eta})$, a contradiction.

The general case can be reduced to the one-dimensional. We leave it as an exercise to the reader. The lemma is proved.

Now we get back to the operator $P[x,u,\alpha]$ given by formulas (18).

Theorem 2. Let a triple $w_0 = (x_0, u_0, \alpha_0)$ be such that the linear operator $P[w_0]$ is onto. Then for any bounded set $S \subset L_\infty^N$ there exist numbers $b > 0,\ \varepsilon > 0$ and a weak-* neighborhood $V(\alpha_0)$ possessing the property:
for any triple $(x,u,\alpha) \in W$ with

$$||x - x_0||_C < \varepsilon, \quad ||u - u_0||_\infty < \varepsilon, \tag{20}$$

$$\text{and} \qquad \alpha \in V(\alpha_0) \cap S, \tag{21}$$

the operator $P[x,u,\alpha]$ is $b-$ covering, i.e.,

$$P[x,u,\alpha]\,(D_1^W) \supset D_b^Z\ .$$

Proof. Consider the operator $\tilde{P}[\alpha] = P[x_0, u_0, \alpha]$ for the fixed (x_0, u_0) and arbitrary $\alpha \in S$. By Lemma 1, the matrices $H_j(t) = g_u'(x_0(t), u_0^j(t), t)$ satisfy the above uniform full rank condition, and so, we are in the conditions of Theorem $1'$. (Here $M = N$, all $\beta^i = \alpha^i$, and $\bar{\alpha}^i$ play the role of additional components of the control.) By Theorem $1'$, $\exists\, b > 0$ and a weak-* neighborhood $V(\alpha_0)$ such that $\forall\, \alpha \in V(\alpha_0) \cap S$ the operator $\tilde{P}[\alpha]$ is $b-$ covering. But, since the functions f, g and their derivatives are equicontinuous in (x,u) and the set S is bounded, the operators $\tilde{P}[\alpha] = P[x_0,u_0,\alpha]$ and $P[x,u,\alpha]$ are close to each other in the operator norm if x,u are uniformly close to x_0, u_0. Hence, $\forall\, b' < b,\ \exists\, \varepsilon > 0$ such that, for all x and u satisfying (20), $P[x,u,\alpha]$ is $b'-$ covering. Theorem 2 is proved.

3. COVERING OF NONLINEAR OPERATORS

Let us now pass from the covering of linear operator P to that of nonlinear operator F. We will use here the following abstract

Theorem 3. (Dmitruk, Milyutin, Osmolovskii, 1980). Let W and Z be Banach spaces, $\mathcal{O}$ an open set in W, let an operator $F : \mathcal{O} \to Z$ be strictly differentiable at all points of a set $\Omega \subset \mathcal{O}$, and $\exists\, b > 0$ such that $\forall\, w \in \Omega$ the operator $F'(w)$ is $b-$ covering. Then $\forall\, b' < b$ there exists an open set $G \supset \Omega$ such that F is $b'-$ covering on G, i.e., for any ball $D_\gamma(w) \subset G$, its image contains the corresponding ball of radius $b'\gamma$:

$$F(D_\gamma(w)) \supset D_{b'\gamma}(F(w)).$$

Also, we will use the following obvious

Lemma 2. Suppose an operator $F: W \to Z$ is $b-$ covering on an open set G for some $b > 0$, there is given a bounded set $\Omega \subset G$, and $\exists\, \delta > 0$ such that $\Omega + D_\delta \subset G$. Let $w_0 \in \Omega$, and consider the level set

$$\mathcal{M} = \{ w \in G \mid F(w) = F(w_0) \}.$$

Then, there exists a constant L such that $\forall\, w \in \Omega$

$$\text{dist}\,(w, \mathcal{M}) \;\le\; L \, ||F(w) - F(w_0)||. \tag{22}$$

This lemma, Theorems 2 and 3 imply the following assertion for our functional space W and the operator F defined by the left hand sides of (17).

Corollary. Let a triple $w_0 = (x_0, u_0, \alpha_0)$ be such that $F'[w_0]$ is onto. Then, for some $\varepsilon > 0$ and some weak-* neighborhood $V(\alpha_0)$, estimate (22) holds for any triple w satisfying (20), (21).

Now, let S be the set of functions taking their values in the standard simplex:

$$S = \; \{ \alpha \in L^N_\infty \mid \; \alpha^i(t) \ge 0, \; \sum \alpha^i(t) = 1 \}.$$

Denote by σ^* the weak-* topology in L^N_∞ .

Theorem 4. Let $F(x_0, u_0, \alpha_0) \,=\, 0$, the operator $F'(x_0, u_0, \alpha_0)$ be onto, and suppose that

$$\forall\, i, \quad \text{vrai}\min_t \; \alpha_0^i(t) > 0, \qquad \text{i.e.,} \;\; \alpha_0 \in \mathit{int}\, S. \tag{23}$$

Then, in any $(C, L_\infty, \sigma^*)-$ neighborhood of the point (x_0, u_0, α_0) there exists a point (x, u, α) still satisfying the equality $F(x, u, \alpha) = 0$, and such that $\forall\, i$ the function $\alpha^i(t)$ takes only two values: 0 or 1 (i.e., all α^i are characteristic functions χ_{E^i} of some measurable sets E^i).

Note that, if one defines the control $u = \sum u^i(t)\chi_{E^i}(t)$, then the pair (x, u) will satisfy, instead of (17), the usual system

$$\begin{gathered} \dot{x} - f(x, u, t) = 0, \\ K(x(0), x(T)) = 0, \\ g(x, u, t) = 0, \end{gathered} \tag{24}$$

that presents the conventional equality type constraints in optimal control problems. It is just in the study of this system one obtains the extended (relaxed, convexified) system (17). Theorem 4 allows thus to approximate a trajectory of the relaxed system (17) involving sliding control modes by trajectories of the initial system (24) with an ordinary control. This fact can be used in proving the Maximum Principle of Pontryagin's type for optimal control problems with state and so-called regular mixed constraints by passing to a system with sliding mode controls. For example, this is a key point in the proofs given in (Chukanov, 1990) and (Dmitruk, 1993).

To prove Theorem 4, one should use the above Corollary, and then follow the arguments of Sec. 5–7 in (Dmitruk, 1976).

ACKNOWLEDGEMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 00-15-96109.

REFERENCES

Dmitruk, A.V. (1976). The justification of the sliding mode method to optimal control problems with mixed constraints. *Functional Analysis and Appl.*, **10**, pp. 197–201.

Dmitruk, A.V., A.A. Milyutin, and N.P. Osmolovskii (1980). Lyusternik's theorem and the theory of extrema. *Russian Math. Surveys*, **35**, no. 6, pp. 11–51. (Translated from *Успехи мат. наук*, 1980, т. 35, N 6, с. 11–46).

Dmitruk, A.V. (1993). Maximum principle for a general optimal control problem with state and regular mixed constraints. *Computational Mathematics and Modeling*, **4**, no. 4, pp. 364–377. (Translated from "Оптимальность управляемых динамических систем", М., ВНИИСИ, 1990, вып. 14, с. 26–42).

Chukanov, S.V. (1990). Maximum Principle for optimal control problems with integral equations. In *"Необходимое условие в оптимальном управлении"* ("Necessary Condition in Optimal Control", in Russian), Ch. 6; Nauka: Moscow.

This paper is published in *"Nonlinear Control Systems 2001"*
(ed. A.B.Kurzhanski and A.L.Fradkov), Elsevier, 2002, vol. 2, p. 1061-1064.