Second Order Optimality Conditions for Singular Extremals

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Abstract

We consider the class of optimal control problems, linear in the control, with control bounded by linear inequalities, and with terminal equality and inequality constraints. Both control and state variables are multidimensional, and the examined control is totally singular.

For such problems we suggest quadratic-order necessary and sufficient conditions for a weak and a so-called Pontryagin minimum, which is a minimum of intermediate type between classic weak and strong minima. Necessary conditions transform into sufficient ones only by strengthening an inequality, what is similar to conditions in the classical analysis and calculus of variations (close pairs of conditions).

Key words: singular extremal, weak and Pontryagin minimum, quadratic order of estimation, necessary and sufficient conditions, third variation of Lagrange function.

1 Statement of the problem

The problem of consideration is:

$$J = \kappa_0(p) \to \min, \quad K(p) = 0, \tag{1}$$

$$\kappa_i(p) \le 0, \quad i = 1, \ldots, \nu, \tag{2}$$

$$\dot{x} = f(x,t) + F(x,t)u, \tag{3}$$

$$u(t) \in U(t). \tag{4}$$

Here $p = (x_0, x_1)$, $x_0 = x(t_0)$, $x_1 = x(t_1)$, the time interval $[t_0, t_1]$ is fixed; x is a Lipschitz function and u is a bounded measurable function, the dimensions of x, u, K are $d(x)$, $d(u)$, $d(K)$ respectively. The system (3) is linear in the control u, but nonlinear in the state variable x.

Assumptions. A1) All functions κ, K, f, F are twice continuously differentiable in x and Lipschitzian in t.

A2) The set $U(t)$ is convex, continuous (in the Hausdorff metric) and uniformly solid in t.

2 Basic notions

We denote by W the space of all pairs of functions $w = (x, u)$ of the abovementioned type. Let $w^0 = (x^0, u^0)$ be an examined trajectory. We assume that $w^0(t)$ is totally singular over the whole $U(t)$, and $u^0(t)$ is continuous.

The Pontryagin Maximum Principle (MP) says that there exist Lagrange multipliers $\alpha = (\alpha_0, \ldots, \alpha_\nu) \geq 0$, $c \in R^{d(K)}$, and a Lipschitz function $\psi(t)$ such that $\mid \alpha \mid + \mid c \mid = 1$, $\dot{\psi} = -H_x$, $\psi(t_0) = l_{x_0}$, $\psi(t_1) = -l_{x_1}$, and (because of the total singularity of w^0) $H_u = 0$, where $l(p) = \alpha \cdot \kappa(p) + c \cdot K(p)$,

$$H(x, u, t) = \psi[f(x, t) + F(x, t)u], \qquad \kappa = (\kappa_0, \ldots, \kappa_\nu).$$

To simplify further considerations we assume here that such a collection $\lambda = (\alpha, c, \psi)$ is unique. (For a general case see [12-16].)

Without loss of generality we take $w^0(t) \equiv 0$, and $\kappa_i(0, 0) = 0 \quad \forall i = 0, 1, \ldots, \nu$, i.e. all indices are active.

It is generally known that the fulfillment of the MP does not guarantee the optimality of the given trajectory. In particular it is true for our class of problems (1)-(4). (One of the first examples - the famous Lawden's spiral.) Therefore, other optimality conditions are desirable, which may be obtained via investigations of the second order. Such investigations for this class of problems (in some special statements) have been done since early 1960-s by many authors: Kelley, Kopp, Moyer, Bryson, Robbins, Goh, Vapnyarsky, Bolonkin, Speyer, Jacobson, Bell, McDanell, Powers, Gabasov, Kirillova, Krener, Agrachiov, Gamkrelidze, Milyutin, Knobloch, Zelikin, Gurman, Dykhta, Lamnabhi-Lagarrigue, Stefani and others (see [2-10] and references therein). An overwhelming majority of works are devoted to higher-order *necessary* conditions which have *a pointwise* character (we call them Legendre conditions). A large number of such conditions have been obtained. But the question have been remained open: what is a full set of necessary conditions? Few works are devoted to *sufficient* conditions, but these conditions obtained till now are rather far from necessary ones (in particular because they include conditions of Frobenius type, which are not conditions of any finite order). We investigate this class of problems to obtain both necessary and close to them sufficient second order conditions of optimality. We don't assume Frobenius type conditions.

First of all, we should define more accurately what means "optimality" and what are "second order" conditions.

Types of minimum. Speaking of optimality, we should point out a type of minimum under consideration. We consider the following two types of minimum - a weak and a so-called Pontryagin minimum (Π-minimum).

A weak minimum, known from the classics, is a minimum in the norm

$$||w|| \;=\; ||x||_\infty + ||u||_\infty,$$

or, in other words, a minimum with respect to uniformly small variations. A Pontryagin minimum includes in addition so-called "needle-type" variations.

Definition. We say that $w^0 \equiv 0$ is *a Pontryagin minimum point* in problem (1)-(4), if for all N there exists an $\varepsilon > 0$ such that w^0 is a minimum point in problem (1)-(4) on the set

$$||x||_\infty < \varepsilon, \quad ||u||_1 < \varepsilon, \quad ||u||_\infty \leq N.$$

In other words, there can exist no sequence $w_n = (x_n, u_n)$ such that

$$||x_n||_\infty \to 0, \quad ||u_n||_1 \to 0, \quad ||u_n||_\infty \leq \mathcal{O}(1), \tag{5}$$

all constraints (1)-(4) are satisfied, and for all n $J(p_n) < J(p^0)$.

We call *Pontryagin sequences* those satisfying (5); the set of all such sequences we denote by Π. A Pontryagin minimum (Π-minimum) obviously occupies an intermediate position between the classic weak and strong minima.

Quadratic order of estimation. Speaking of "second order" conditions, to be more accurate, we should point out a quadratic functional of estimation, regarding to which all the functions in the problem to be considered. For example, in the classical calculus of variations (where $\dot{x} = u$) the appropriate functional is

$$\gamma_0(w) = |\, x(t_0) \,|^2 + \int |\, u(t) \,|^2 \, dt. \tag{6}$$

(here and throughout the paper all integrals are taken over the whole interval $[t_0, t_1]$). This functional is adequate also for optimal control problems nonlinear in the control (see [11]), but obviously it is too rough for the problem (1)-(4), since the last is linear in u. It turns out that the adequate quadratic functional of estimation for this class of problems is

$$\gamma(w) = |\, x(t_0) \,|^2 + |\, y(t_1) \,|^2 + \int |\, y(t) \,|^2 \, dt, \tag{7}$$

where

$$\dot{y} = u, \qquad y(t_0) = 0 \tag{8}$$

(the latter is preserved throughout the paper). Note that the control u does not come as such in the quadratic order (9); it comes only through the new state variable y. Here we give conditions of this order γ.

Consider *Lagrange function*

$$\Phi(w) = l(p) + \int ((\psi, \dot{x}) - H(x, u, t))dt, \tag{9}$$

and its second variation - the quadratic functional

$$\Omega(w) = (l''p, p) - \int ((H_{xx}x, x) + 2(x, H_{xu}u))dt. \tag{10}$$

Define the matrices $A(t) = f_x(0, t)$, $B(t) = F(0, t)$ and the tensor $R(t) = F_x(0, t)$ in such a way that the equation (3) is reduced to

$$\dot{x} = A(t)x + B(t)u + (R(t)x, u) + \text{h.o.t.} \tag{11}$$

Let $\mathcal{K}$ be the so-called critical cone, consisting of all $w = (x, u)$ in W such that $\kappa'(0,0)p \le 0, \quad K'(0,0)p = 0$, and

$$\dot{x} = A(t)x + B(t)u. \tag{12}$$

Recall the well-known *Goh transformation*: $(x, u) \to (\xi, y, u)$, where $\xi = x - By$, and hence

$$\dot{\xi} = A\xi + B_1 y, \quad B_1 = AB - \dot{B}. \tag{13}$$

Under this transformation the functional (10) takes the form:

$$\Omega(\xi, y, u) \;=\; g(\xi_0, \xi_1, y_1) \;+ \int ((D\xi, \xi) + (P\xi, y) + (Qy, y) + (Vy, u))dt, \tag{14}$$

where g is a terminal quadratic form, $Q(t)$ is a symmetric and $V(t)$ is a skew-symmetric Lipschitz matrices.

We also recall Goh conditions (firstly proposed in [3]):

$$\forall t \qquad a)\, V(t) = 0, \quad \text{and} \quad b)\, Q(t) \ge 0. \tag{15}$$

which obviously have a pointwise character.

Now we pass to optimality conditions obtained.

3 Conditions of a weak minimum

Consider firstly the case when $u^0(t)$ goes strictly inside $U(t)$, i.e. for some $\varepsilon > 0$, for every t the ε -neighborhood of $u^0(t)$ is contained in $U(t)$. It is clear that in this case constraint (4) is not essential, so we can neglect it.

Theorem 1 [2, 12]. *a) Let w^0 be a weak minimum point in problem (1)-(3). Then Goh conditions (15) hold, and moreover*

$$\Omega(w) \ge 0 \quad \text{for all } w \in \mathcal{K}. \tag{16}$$

b) Suppose that Goh conditions (15) hold, and for some $a > 0$

$$\Omega(w) \ge a\gamma(w) \quad \text{for all } w \in \mathcal{K}. \tag{17}$$

Then w^0 is a weak minimum point in problem (1)-(3).

As one can see, these necessary and sufficient conditions are close each to other; we call them a close pair of conditions. In this sense these conditions are precisely analogous to those in the analysis and the calculus of variations.

It is worth to note here (and it is known from the classics) that the full set of necessary conditions must definitely contain an inequality of the form (16), which is non-Legendre, and just by the strengthening of which necessary conditions transform into sufficient ones.

Now let us consider the general case, i.e. when $u^0(t)$ may contact the boundary of $U(t)$. Here we take some assumptions about the character of contacts:

B1) In a neighbourhood of contacts the control set $U(t)$ is a polyhedron.

B2) As before, the examined extremal $w^0(t)$ is totally singular with respect to the whole $U(t)$.

B3) Contacts with the bound of $U(t)$ have in some sense "good" character.

The critical cone in this case is $\mathcal{R} = \mathcal{K} \cap \mathcal{N}$, where the cone

$$\mathcal{N} = \{\bar{w} \in W: \ \bar{u}(t) \in N(t) = con(U(t) - u^0(t))\}$$

is generated by the pointwise cone $N(t)$.

Here Goh conditions must be regarded with respect to the maximal linear subspace $l(t)$ in $N(t)$, i.e. if $P(t)$ is the projector onto $l(t)$, then

$$\forall t \qquad a)\, PVP(t) = 0, \qquad b)\, PQP(t) \geq 0. \tag{18}$$

Theorem 1 is still valid for this case, if we replace Goh conditions (15) by (18). This was proved by A.A.Milyutin but is yet unpublished.

4 Pontryagin minimum for a free control

Assume as before that $u^0(t)$ goes strictly inside $U(t)$, but now let us consider a Π-minimum. For a Π-minimum constraint (4) is essential and we cannot neglect it. Consider firstly the case, when (4) is absent, and so u is unbounded.

It is well known, that the first order conditions both for a weak and a Π-minimum are one and the same (for any problem convex in the control). Let us now pose the question: will conditions of the above- stated quadratic order γ be strengthened if we pass from a weak to a Π-minimum? Or in other words - *do Pontryagin (e.g. needle-type) variations bring some new optimality condition in addition to those provided by uniformly small variations?* The answer is that they do.

To give an accurate formulation, define the cubic functional

$$\rho(w) = \int [-(H_{uxx}x, x, u) + 2((Rx, u), H_{xu}y)]dt. \tag{19}$$

It is the third variation of the Lagrange function (9) at zero (in W) on equation (3) to within $o(\gamma)$ on Pontryagin sequences, see [13, 14].

Using the Goh transformation we reduce ρ to the form

$$\rho(w) = \int ((T_1\xi, \xi, u) + (T_2\xi, y, u) + (\mathcal{E}y, y, u))dt. \tag{20}$$

Here the essential part is presented by the last term [14].

For all t^* we introduce the differential 1-form

$$\omega(t^*) = \ (\mathcal{E}(t^*)y, y, dy) \ = \sum_{ijk} \mathcal{E}_{ijk}(t^*) y^i y^j dy^k. \tag{21}$$

The new pointwise condition, provided by Pontryagin variations, is: for every t^* 1-form (21) is closed, i.e.

$$d\omega(t^*) \ = \sum_{ijk} \mathcal{E}_{ijk}(t^*)(y^i dy^j + y^j dy^i) \wedge dy^k = 0 \tag{22}$$

(here t^* is a parameter, the differential is taken with respect to y).

Theorem 2. *a) Let w^0 be a Pontryagin minimum point in problem (1)-(3). Then both Goh conditions (15) and the new condition (22) hold, and as before, inequality (16) is valid.*

b) Suppose that both Goh conditions (15) and the new condition (22) hold, and as before, for some $a > 0$ inequality (17) is valid. Then w^0 is a Pontryagin minimum point in problem (1)-(3).

The proof is given in [14] and is based on a general theory of higher order conditions, developed recently by A.A.Milyutin and his co-workers [1].

5 Pontryagin minimum for a bounded control

Now let the constraint (4) be present, but as before $u^0(t)$ goes strictly inside $U(t)$. In this case we must replace condition (22) by another one.

Consider the functional

$$L(y) = \int ((Qy, y) + (Vy, u) + (\mathcal{E}y, y, u))dt, \tag{26}$$

where the two first terms are from (14) and the last one is from (20). The new condition is: $V(t) \equiv 0$, and for every t^* and every Lipschitz function $y(t)$, having $y(t_0) = y(t_1) = 0$ (we call such a function a cycle) and $\dot{y} = u \in U(t^*)$, the following inequality holds:

$$L[t^*](y) = \int ((Q(t^*)y, y) + (\mathcal{E}(t^*)y, y, u))dt \geq a \int (y, y)dt, \tag{27}$$

where a is a real number. The functional in (27) is got by freezing the coefficients of (26) at the point t^* (with accounting that $V(t)$ vanishes) the set $U(t)$ is also frozen at t^*.

Theorem 3 [13, 16]. *a) Let w^0 be a Pontryagin minimum point in problem (1)-(4). Then both Goh conditions (15) and condition (27) with $a = 0$ hold, and as before, inequality (16) is valid.*

b) Suppose that Goh conditions (15) hold, and for some $a > 0$ inequalities (27) and (17) are valid. Then w^0 is a Pontryagin minimum point in problem (1)-(4).

The proof is based on the general theory [1] and yet unpublished.

Condition (27) has a pointwise character, so it can be regarded as a new Legendre type condition. As one can see, it concerns not only the second variation of the Lagrange function, as usual, but the sum of the second and the third variations. In the case when $U(t) \equiv R^{d(u)}$, condition (27) with $a = 0$ decomposes onto (15,b) and (22).

But for an arbitrary convex $U(t^*)$ containing the origin in its interior, (27) does not decompose, and we get an auxiliary problem: to determine all the $a \in R$ such that (27) holds for any cycle with $\dot{y} = u \in U(t^*)$. This problem has intrinsic interest, and it seems rather difficult. However, at the present time its solution is known for three cases, when U is: a) the whole space (see above), b) a stripe $c \leq (m, u) \leq d$, where m is an arbitrary vector in $R^{d(u)}$ (A.A.Milyutin), c) an arbitrary ellipsis on the plane (A.A.Milyutin and the author).

6 Pontryagin minimum for $u^0(t)$ contacting the bound of U(t)

As before, conditions for a Π-minimum are similar to those for a weak minimum, but instead of (27) a new condition must be introduced.

Denote by Leg $\Pi(U)$ the set of all Pontryagin sequences $w_n = (x_n, y_n, u_n)$, satisfying (4, 8, 12) and such that

$$| x_n(0) | + | y_n(1) | + \int | y_n(t) | \, dt \ = o(\sqrt{\gamma_n}). \tag{29}$$

We call them Legendre sequences. A characteristic example is $y(t)$, having a triangle shape, based on an interval, tending to a point t^*.

Let $a \in R$ and for any sequence from Leg $\Pi(U)$ the functional (26) satisfies inequality:

$$\liminf_{n\to\infty} \frac{L[\lambda](w_n)}{\gamma(w_n)} \geq a. \tag{30}$$

If $u^0(t)$ goes strictly inside $U(t)$, this condition is reduced to (27).

Theorem 4 [16]. *a) Let w^0 be a Pontryagin minimum point in problem (1)-(4). Then both Goh conditions (28) and condition (30) with $a = 0$ hold, and as before, inequality (16) is valid.*

b) Suppose that Goh conditions (28) hold, and for some $a > 0$ inequalities (30) and (17) are valid. Then w^0 is a Pontryagin minimum point in problem (1)-(4).

A more convenient form of condition (30) will be produced in the nearest papers by the author.

7 Examples

Example 1. $\quad \dot{x}_1 = u_1, \quad \dot{x}_2 = u_2 - bx_2u_1, \qquad x(0) = 0, \quad t \in [0,1]$,
the control u is unbounded, the functional is

$$J = \int (2x_1u_1 + 2x_2u_2 + x_1^2 + x_2^2)dt \ \to \min,$$

and $x^0 \equiv u^0 \equiv 0$. Here the critical cone $\mathcal{K}$ is given by $\dot{x} = u, \quad x(0) = 0$. The second variation of Lagrange function is $\Omega[\lambda] = J$ and is equal to γ on $\mathcal{K}$, where

$$\gamma(y) = \ \gamma(y_1) \ + \gamma(y_2) \ = | y_1(1) |^2 + | y_2(1) |^2 + \int (| y_1(t) |^2 + | y_2(t) |^2)dt;$$

$$\rho[\lambda](w) = \ 2b \int y_2^2 u_1 dt \ + \ o(\gamma),$$

$$\omega[\lambda] = 2by_2^2 dy_1, \qquad d\omega[\lambda] = -4by_2 dy_1 \wedge dy_2.$$

By Theorem 1 for each b there is a weak minimum at zero, but according to (22) and to Theorem 2 (a), only for $b = 0$ there is a Pontryagin minimum.

Now let us take $b \neq 0$ and add the constraint $|\, u \,| \leq r$ (a circle) to the problem. For which r there will be a Π-minimum? We know that since there is a weak minimum, obviously there is also a Π -minimum for all sufficiently small r. Theorem 3 allows us to determine the precise critical value of such r.

According to this Theorem, to answer above question we must only check condition (27). Here the result is: $\max\, a = 1 - r \,|\, b \,|$, so

if $\quad r \,|\, b \,| < 1, \quad$ then a Π-minimum holds,

and if $\quad r \,|\, b \,| > 1, \quad$ then a Π-minimum fails.

Consider the initial problem with another constraint: $|\, u_1 \,| \leq r$ (a stripe). Here it can be shown that condition (27) holds for all $a \leq 1 - 2r \,|\, b \,|$, so

if $\quad 2r \,|\, b \,| < 1, \quad$ then a Π-minimum holds,

and if $\quad 2r \,|\, b \,| > 1, \quad$ then a Π-minimum fails.

Note that the critical value of r for a stripe is two times less than that for a circle.

Let us now consider an ellipsis:

$$\left(\frac{u_1}{r_1}\right)^2 + \left(\frac{u_2}{r_2}\right)^2 \leq 1.$$

Here the critical value for a Π- minimum is: $2\frac{r_1 r_2}{r_1 + r_2} \,|\, b \,| = 1.$

Note that if $r_1 = r_2$, it is reduced to the case of a circle, and if $r_2 \to \infty$, we get precisely the critical value for a stripe.

Consider yet another constraint to initial problem: $u_1 \geq 0$. In this case the control u^0 lies entirely on the boundary of $U(t)$.

Let us check the new Legendre condition (30). Here it means that for any sequence of cycles $y^{(n)}$, having $\dot{y}_1^{(n)} = u_1^{(n)} \geq 0$ (in below we omit the index n), and such that

$$|\, y_1(1) \,| + |\, y_2(1) \,| + \int (|\, y_1(t) \,| + |\, y_2(t) \,|) dt \;=\; o(\sqrt{\gamma(y)}), \tag{31}$$

the following inequality should be valid:

$$L = \int (y_1^2 + y_2^2 + 2b y_2^2 u_1) dt \;\geq\; (a - o(1)) \cdot \gamma(y). \tag{32}$$

Note that since $u_1 \geq 0$, i.e. y_1 is monotone, for all t $y_1(t) \leq y_1(1)$, and hence we can take $\gamma(y_1) = |\, y_1(1) \,|^2$. Then (31) implies that

$$\gamma(y_1) \;=\; o(\gamma(y_2)), \tag{33}$$

and condition (32) takes the form:

$$\int (y_2^2 + 2b y_2^2 u_1) dt \;\geq\; (a - o(1)) \int y_2^2 dt. \tag{34}$$

Taking here the cubic term by parts, we get regarding (33) that

$$|\int y_2^2 u_1 dt \,| = |\; - \int y_1 y_2 u_2 dt \,| \leq$$

$$\leq \|y_1\|_2 \cdot \|y_2\|_2 \cdot \|u_2\|_\infty \;\leq\; o(\sqrt{\gamma(y_2)}) \cdot \sqrt{\gamma(y_2)} \cdot \mathcal{O}(1) \;=\; o(\gamma(y_2)),$$

hence (34) is valid with $a = 1$, and by Theorem 4 there is a Π-minimum at w^0.

Example 2. Bilinear system:

$$\dot{x} = p + uAx + vBx, \qquad x(0) = q, \tag{35}$$

$$J = (l, x(T)) \to \max, \qquad \dim x = n,$$

the controls u, v are scalars and the interval $[0, T]$ is fixed.

Let $u^0 = v^0 = 0, \quad x^0 = q + pt$. Then MP yields

$$lAp = lAq = lBp = lBq = O. \tag{36}$$

Assume it holds. Set $\dot{y} = u,\ y(0) = 0,\ \dot{z} = v,\ z(0) = 0, \quad w = (y, z), \quad r = (u, v),$ so $\dot{w} = r$. Define matrices

$$P = \begin{pmatrix} lA^2p & lBAp \\ lABp & lB^2p \end{pmatrix}, \qquad Q = \begin{pmatrix} lA^2q & lBAq \\ lABq & lB^2q \end{pmatrix}.$$

Goh equality condition $(15, a)$ yields:

$$l(AB - BA)p = l(AB - BA)q = 0. \tag{37}$$

Assume it also holds (otherwise there are no even a weak minimum), which implies that matrices P and Q are symmetric. The second variation of Lagrange function is:

$$\Omega = \frac{1}{2}((Q + PT)w(T), w(T)) + \int (Pw(T), w)dt - \frac{1}{2}\int (Pw, w)dt.$$

Goh inequality condition $(15, b)$ means:

$P \le 0$ (necessary), $\quad P < 0$ (sufficient).

Assume that $P < 0$. Let ρ_1, ρ_2 and θ_1, θ_2 are the eigenvalues of $-P$ and $-Q$ respectively.

Result: if $\theta_1 < 0$ or $\theta_2 < 0$, then $\Omega < 0$ and by Theorem 1 there are no a weak minimum at the given extremal; if $\theta_1 > 0$ and $\theta_2 > 0$, i.e. $Q < 0$, then (17) is fulfilled and there is a weak minimum.

To analyse a Π-minimum assume that $P < 0$ and $Q < 0$.

Result: if

$$l[A, [B, A]]p = l[B, [B, A]]p,$$

$$l[A, [B, A]]q = l[B, [B, A]]q, \tag{40}$$

(where [,] denotes a commutator) then condition (22) holds and by Theorem 2 there is a Π-minimum for unbounded u, v.

Suppose that conditions (40) are not valid. Consider additional constraint: $-b \le u \le a$. Here Legendre function (26) is:

$$L = \int [G(y, z) + M(y, z)u]dt,$$

where $\quad G = -\frac{1}{2} l(yA + zB)(yA + zB)p, \quad$ and

$$M(y,z) = l[A,[A,B]]x^0(t)yz \;+\; \frac{1}{2} l[B,[A,B]]x^0(t)z^2.$$

Using the abovementioned result by A.A.Milyutin we have to consider a quadratic form of three variables:

$$\Phi = a^{-1}G(y,z_1) + M(y,z_1) + b^{-1}G(y,z_2) - M(y,z_2). \tag{42}$$

Result: by Theorem 3 a Π-minimum implies that $\Phi \geq 0$, and $\Phi > 0$ implies that there is a Π-minimum at the examined extremal.

Thus, we have reduced the problem to a standard question of linear algebra: to check the nonnegativity and positivity of quadratic form (42). Another additional constraint, where (u,v) belong to an ellipsis, can also be considered in this way.

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