MAXIMUM PRINCIPLE FOR THE GENERAL OPTIMAL CONTROL PROBLEM WITH PHASE AND REGULAR MIXED CONSTRAINTS

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We prove the maximum principle for the problem outlined in the title with the aid (or more precisely in the class) of so-called sliding variations.

INTRODUCTION

The by-now classical maximum principle for the simplest nonlinear optimal control problem was formulated by Pontryagin and proved by Boltyanskii over 30 years ago (see [1, 2, 3] and also [8]). Since then, the maximum principle has been derived for various optimal control problems in an enormous number of studies. This is well-known and we do not give a full list of references here; we only mention [4-7, 9-24, 27-30]. In our view, the deepest and most thorough studies are those of Milyutin and Dubovitskii [9-14]. These authors advance very far in their development of the maximum principle for problems with ordinary differential equations – first to problems with phase constraints [9]* and then to the general problem with phase and arbitrary (in general, nonregular) mixed constraints [10-14]. These authors completely resolve the issue of deriving the necessary first-order conditions for these problems.

Despite these achievements, we are still witnessing the publication of studies that derive the maximum principle for various problems which are particular cases of the general Dubovitskii–Milyutin problem. The authors of some of these studies possibly do not realize that they are rederiving well-established results.**

This is partly due to the fact that, on the one hand, optimal control results are primarily interesting for applied engineers, while the references [9-14] are far from being accessible to every engineer, because they use a fairly complex mathematical apparatus which is not familiar to engineers; on the other hand, optimal control theory as such so far has failed to attract a sufficiently wide followign of mathematicians. "Optimizers" still feel that if a problem originates in engineering practice, then its solution (and preferably the process of its analysis) should remain in the domain of concepts which are familiar in the engineering environment (compare this situation with the totally different state of things in the theory of partial differential equations or in probability theory).

This, in our view, accounts for repeated attempts to simplify the proof of the maximum principle. However, even if some studies achieve a certain simplification of the proof, this is the result of extreme simplification of the problem, and not an achievement of the particular method of analysis. The positive impact of these attempts is that they highlight the complexity of the problem. They have shown that the problem cannot be solved by elementary techniques, without invoking new methods (specifically, methods of functional analysis). We are thus forced to admit that the complexity of the work of Dubovitskii–Milyutin is not contrived: it is inherent to the complexity of the underlying problem.

Yet there is a fairly wide class of problems for which the maximum principle can be derived relatively simply. These are the problems with pure phase constraints and so-called "regular" mixed constraints. They cover most of the problems considered in the literature. The maximum principle for these problems was derived by Milyutin and Dubovitskii back in the 1960s, immediately after their derivation of the maximum principle for problems with phase constraints [9]. However, it has never been published because, in the authors' opinion, the addition of regular mixed constraints to phase constraints did not constitute a significant advance.

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*An earlier study by Gamkrelidze [4] provides only a partial answer: for instance, it does not determine the sign of the discontinuity of the conjugate variable ψ at the points where the phase constraint is reached.

**This remark does not apply to the cited references.

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Cumulative experience shows that the derivation of the maximum principle for these problems is of considerable interest for a broad class of users. This view is confirmed by the publication of many works dealing with particular cases of these problems. The paper by Neustadt and Makowsky [18] is the only study that derives the maximum principle for the general "regular" problem, but it contains many technical complications and can hardly be viewed as transparent and easily accessible. In any event, the derivation of the maximum principle for these problems is not covered in the current textbooks, which is a definite gap.

In this paper, we prove the maximum principle for the general "regular" problem using so-called sliding modes [5, 6]. The idea of using sliding modes as the class of variations (i.e., sliding variations) to prove the maximum principle was originally proposed by Milyutin. He also outlined the entire scheme of the proof, which we closely follow (only the proof of Theorem 2 deviates from Milyutin's scheme). This proof is fairly transparent, and almost all arguments are on the level of well-known results of functional analysis. It may therefore be also interesting for mathematicians (e.g., used as a special course for university students).

The main points of the proof are the following:

a) the condition of stationarity in the class of small variations (local maximum principle);

b) Lemma 1 on absence of singularities;

c) the associated problem;

d) Theorem 2 on "confidence" in equations of the associated problem (on correctness of these equations, or else the approximation theorem);

e) Theorem 3 on stationarity in the associated problem;

f) the finite-valued maximum principle;

g) the global maximum principle.

Only two of these points require special attention to technical detail. These are Lemma 1 and Theorem 2. In all other respects, the proof can be read without special effort. Also note the elegant technique used in the proof to pass from finite-valued to global maximum principle: this is accomplished by using a centered system of compacta, an idea first proposed by Dubovitskii and Milyutin. Another fairly interesting result is Lemma 2 on the L_1-distance to an "almost extreme" point of some set of functions.

The first example known to us in which sliding variations are used to prove the maximum principle (for some simple, so-called Lyapunov, problem) is provided by [9]; more recently this method was applied (in different versions and for different problems) in [7, 14-18, 20-22, 28] and elsewhere.

Finally note that the maximum principle for a regular (although not general) problem can be derived by an alternative technique, replacing the sliding variations with so-called ν-variations (or more precisely in the class of ν-variations) proposed by Dubovitskii and Milyutin [9, 14]. This technique was presented by Milyutin in his lectures to fourth-year students of the Faculty of Mechanics and Mathematics of Moscow State University in 1972-1973 and has recently been published in [27].

I am grateful to A. A. Milyutin for numerous discussions and for clarification of various issues considered in this paper.

1. THE MAIN PROBLEM

We consider the following optimal control problem on a fixed time interval $T = [t_0, t_1]$.

Problem A:

$$\left\{\begin{array}{ll} \dot{x} = f(x, u, w, t), & (1) \\ \Phi_s(x, t) \leqslant 0, \quad s = 1, \ldots, d(\Phi), & (2) \\ g(x, u, w, t) = 0, & (3) \\ G_j(x, u, w, t) \leqslant 0, \quad j = 1, \ldots, d(G), & (4) \\ w(t) \in W(t) \subset R^{d(w)}, & (5) \end{array}\right.$$

$$\begin{cases} K(x_0, x_1) = 0, & (6) \\ \varphi_i(x_0, x_1) \leqslant 0, \quad i = 1, \ldots, d(\varphi), & (7) \\ J = \varphi_0(x_0, x_1) \to \min. & (8) \end{cases}$$

Here $x_0 = x(t_0)$, $x_1 = x(t_1)$, $d(a)$ is the dimension of the vector a. The functions f, g, x, u, w, K are appropriately dimensioned vector functions; $d(x) = n$, $d(u) = r$, $d(w) = r'$.

The phase variable x is an absolutely continuous function; a control is a pair (u, w) from $L_\infty^{r+r'}$. In this problem, two controls u, w are applied to the controlled system (1) or, in other words, all control parameters are divided into two groups (the second group may be missing*). The problem includes phase constraints (2), mixed equality and inequality constraints (3), (4), a nonfunctional constraint on the second control (5), and unseparated equality and inequality constraints on the trajectory end points (6), (7). The functional (8) also arbitrarily depends on the trajectory end points.

Problem A in this form was introduced by Dubovitskii and Milyutin, who called it canonical problem (see [14, p. 35]**). Many other problems are reducible to the canonical form (for instance, problems with an integral functional, integral equality and inequality constraints; for some other examples, see [14]), but we do not consider them here.

Assumptions

A1) φ and K are continuously differentiable;

A2) Φ is differentiable with respect to x, Φ and Φ_x' are continuous in (x, t);

A3) f, g, G are continuously differentiable in x, u; these functions and their derivatives with respect to x, u are continuous in w and measurable in t;

A4) the set $W(t)$ is arbitrary (even not necessarily measurable in t);

A5) for any $x(t)$, $u(t)$, $w(t)$ that satisfy (4), $\forall j\ \exists\delta > 0$ such that on the set

$$M_j^\delta = \{t \mid G_j(x(t), u(t), w(t), t) > -\delta\}$$

almost everywhere we have $|G_{jx}'| + |G_{ju}'| \geq \text{const} > 0$.

A6) *Regularity of mixed constraints:*

For any triple $x(t)$, $u(t)$, $w(t)$ that satisfies (3), (4) the system of two sets of gradients $\{g_{iu}'(t),\ i = 1, \ldots, d(g)\}$, $\{G_{ju}', j = 1, \ldots, d(G)\}$ is uniformly in t linearly positively independent with respect to the system of sets $\{M_{ij}^\delta\}$ for some $\delta > 0$.

Definition 0. A system of two sets of vectors $\{A_i,\ i = 1, \ldots, l\}$ and $\{B_j, j = 1, \ldots, n\}$ in the space R^r is called linearly positively independent if there are no numbers m_i, $h_j \geq 0$ such that $\Sigma|m_i| + \Sigma h_j > 0$ and $\Sigma m_i A_i + \Sigma h_j B_j = 0$.

Definition 1. A system of two sets of measurable bounded vector functions

$$\{A_i(t), i=1, \ldots, l\} \text{ and } \{B_j(t), j=1, \ldots, n\} \tag{10}$$

in the space R^r is called uniformly in t linearly positively independent (ULPI) on a measurable set $\mathscr{E}$ if there exists $d > 0$ such that for any p_i, $q_i \in L_\infty(\mathscr{E})$, $q_j \geq 0$ such that a.e. on $\mathscr{E}$

$$\sum_i |p_i(t)| + \sum_j q_j(t) = 1, \tag{11}$$

we have a.e. on $\mathscr{E}$ the inequality

$$\left|\sum_i p_i(t) A_i(t) + \sum_j q_j(t) B_j(t)\right| \geqslant d. \tag{12}$$

We give two equivalent definitions that are useful for our purposes.

Definition 1a. System (10) is called ULPI on $\mathscr{E}$ if the following two conditions are satisfied:

*This partition is justified, because the problem is smooth in u and we differentiate with respect to u; smoothness in w is not assumed, differentiation with respect to w is prohibited, but w is subject to a nonfunctional constraint (5).

**Note that the pure phase inequality constraints are not separated from the general mixed constraints in [14]; here we separate them explicitly, because the mixed constraints are required to satisfy certain "regularity" conditions.

1) there exists a function $\bar{u} \in L_\infty{}^r$ such that a.e. on $\mathscr{E}$

$$\forall i\ (A_i(t), \bar{u}(t))=0 \text{ and } \forall j\ (B_j(t), \bar{u}(t))\geqslant 1;$$

2) for all i_0 there exists a function $\bar{u}_0 \in L_\infty{}^r$ such that a.e. on $\mathscr{E}$

$$(A_{i_0}(t), \bar{u}_0(t))=1, \text{ and } \forall i\neq i_0\ (A_i(t), \bar{u}_0(t))=0.$$

Geometrically condition 1 implies that the convex hull of the vectors $B_j(t)$ is separated by a nonzero distance on $\mathscr{E}$ from the subspace $L(t) = \mathrm{Lin}\{A_i(t), i = 1, \ldots, l\}$. Condition 2 implies that the vector $A_{i0}(t)$ is at a nonzero distance from the subspace $L_{i0}(t) = \mathrm{Lin}\{A_i(t), i \neq i_0\}$.

Definition 1b. The system (10) is called ULPI on $\mathscr{E}$ if $\exists d > 0$ such that

1) for any $q_j \in L_\infty(\mathscr{E})$, $q_j(t) \geq 0$ such that $\Sigma q_j(t) = 1$ a.e. on $\mathscr{E}$ we have a.e. on $\mathscr{E}$ the inequality $\rho(\Sigma q_j(t)B_j(t_0), L(t)) \geq d$,

2) for all i_0 a.e. on $\mathscr{E}$ we have $\rho(A_{i0}(t), L_{i0}(t)) \geq d$.

Now assume that in system (10) each vector $B_j(t)$ is considered only on its set M_j.

Definition 2. The system (10) is called ULPI with respect to the system of measurable sets $\{M_j, j = 1, \ldots, n\}$ if there exists $d > 0$ such that for all p_i, $q_j \in L_\infty$ for which $q_t(t) \geq 0$ and which are concentrated on their own M_j inequality (12) is satisfied a.e. on every set $\mathscr{E}$ where (11) is a.e. satisfied.

We give two equivalent definitions of this property.

Suppose that all possible intersections of positive measure $\mathscr{E}_1, \ldots, \mathscr{E}_\nu$ have been formed from given sets $M_1,\ldots,M_n$ and their complements $M_1', \ldots, M_n'$. Denote $J_k = \{j | M_j \supset \mathscr{E}_k\}$.

Definition 2a. The system (10) is called ULPI with respect to the system of sets $\{M_1, \ldots, M_n\}$ if for all k the system $\{A_i(t), i = 1, \ldots, l\}$, $\{B_j(t), j \in J_k\}$ is ULPI on the set $\mathscr{E}_k$.

Definition 2b. The system (10) is called ULPI with respect to the system of sets $\{M_1, \ldots, M_n\}$ if the following two conditions are satisfied:

1) there exists a function $\bar{u} \in L_\infty{}^r$ such that for all i a.e. on T we have $(A_i(t), \bar{u}(t)) = 0$ and for all j a.e. on M_j we have $(B_j(t), \bar{u}(t)) \geq 1$;

2) for every i_0 there exists a function $\bar{u}_0 \in L_\infty{}^r$ such that a.e. on T

$$(A_{i_0}(t), \bar{u}_0(t))=1 \text{ and } \forall i\neq i_0\ (A_i(t), \bar{u}_0(t))=0.$$

The proof of equivalence of Definitions 1, 1a, 1b and 2, 2a, 2b is left to the reader. Note, in particular, that assumption A5 follows easily from A6, but for convenience it has been stated as a separate assumption.

We have thus formulated problem A. Our goal is to establish necessary conditions of "optimality" in this problem.

Let us first specialize the sense of "optimality". This is not an entirely clear notion, although it is used as a matter of course without any explanation. In fact, optimality is always understood in the sense of a minimum in some class of variations, which must be rigorously specified in each particular case. If we are interested in a weak minimum, i.e., a minimum in the class of small variations, then the necessary conditions in problem A (without the control w) are obtained relatively easily (see below Secs. 2, 3). Our task is to extend the class of variations so that it contains so-called spike variations.

Pontryagin variations of the control u are variations such that $\|u - u^0\|_\infty$ is bounded and $\|u - u^0\|_1 \to 0$ (and the same for w).

The class of Pontryagin variations obviously includes the class of small variations and spike variations (providing a natural generalization of the latter). A minimum in the class of Pontryagin variations is called a *Pontryagin minimum*; this is the object of our study. The Pontryagin minimum is obviously intermediate between the weak and the strong minimum.

To derive necessary conditions of Pontryagin minimum in problem A, we consider a family of associated problems for which the stationarity condition is a necessary condition of weak minimum. These conditions are then "compressed" into a condition for problem A, which provides our final result.

2. AUXILIARY PROBLEM. LOCAL MAXIMUM PRINCIPLE ON THIS PROBLEM

We first consider an auxiliary problem and derive the necessary condition of a weak minimum for it, or more precisely the necessary condition of optimality, i.e., the local maximum principle.*

Problem B:

$$
\begin{array}{c|lll}
a_0 & J=\varphi_0(x_0, x_1)\to \min, & & \\
a_i & \varphi_i(x_0, x_1)\leqslant 0, & & \\
c & K(x_0, x_1)=0 & (\nu) & (14)\\
\psi & \dot{x}-f(x, u, t)=0; & (\xi) & (15)\\
\mu_s & \Phi_s(x, t)\leqslant 0; & & (16)\\
m_i & g_i(x, u, t)=0. & (\eta_i) & (17)\\
h_j & G_j(x, u, t)\leqslant 0 & & (18)\\
\beta & (p(t), u(t))-\dot{q}(t)=0 & (\xi) & (19)\\
\sigma_k & (Q_k(t), u(t))\leqslant 0 & &
\end{array}
$$

Here (14), (15) are vector equalities; all others are scalar equalities. The corresponding Lagrange multipliers are noted on the left of each equality.

Assumptions:

B1) The vectors p, $Q_k \in L_\infty{}^r$, $q \in L_\infty{}^1$; the other functions are as in problem A (the control w is absent).

B2) The mixed constraints (16)-(19) are regular, i.e., for any $x(t)$, $u(t)$ satisfying these constraints the sets of vector functions

$$\{g'_{iu}(t),\ p(t)\} \text{ and } \{G'_{ju}(t),\ Q_k(t)\}$$

are ULPI with respect to the system of sets $\{M_{ij}^{\delta}\}$ and

$$N_k^{\delta}=\{t\,|\,(Q_k(t), u(t))\geqslant -\delta\}.$$

Let $x^0(t)$, $u^0(t)$ be a given pair in problem B.

Definition 3. We say that the *Lyusternik condition* is satisfied in problem B at the point (x^0, u^0) if the derivative $P'(x^0, u^0)$ of the mapping

$$P\colon W_1^{1,n}\times L_\infty^r\to L_1^n\times L_\infty^{d(g)}\times L_\infty^1\times R^{d(K)},$$

where $P(x, u) = (\xi, \eta, \zeta, \nu)$ are the left-hand sides of equalities (15), (16), (18), (14), maps $P'(x^0, u^0)$ "onto".

LOCAL MAXIMUM PRINCIPLE FOR PROBLEM B

There exists a nonzero tuple $\lambda = (a, c, \psi, \mu, m, h, \beta, \sigma)$, where $a \in R^{d(\varphi)+1}$, $a \geq 0$, $c \in R^{d(K)}$, ψ is a function of bounded variation, μ_s are nondecreasing left-continuous functions, $\mu_s(t_0) = 0$, $m, h, \beta, \sigma \in L_1$, $h(t) \geq 0$, $\sigma(t) \geq 0$, such that on the given trajectory (x^0, u^0) the functions

$$l=(a, \varphi)+(c, K),\quad \varphi=(\varphi_0, \varphi_1, \ldots, \varphi_{d(\varphi)}),$$

$$\overline{H}=(\psi, f)-(\dot{\mu}, \Phi)-(m, g)-\beta[(p, u)-\dot{q}]-(h, G)-(\sigma, Qu).$$

satisfy the equalities

$$-\dot{\psi}=\overline{H}_x',\quad \overline{H}_u'=0,$$

$$\psi_0=l'_{x_0},\quad \psi_1=-l'_{x_1} \tag{20}$$

and moreover the complementary slackness conditions are satisfied:

$$
\left.\begin{array}{l}
a_i\varphi_i(x_0^0, x_1^0),\quad \forall i\geqslant 1,\\
\dot{\mu}_s(t)\Phi_s(x^0(t), t)\equiv 0,\quad \forall s,\\
\text{a.e. } h_j(t)G_j(x^0(t), u^0(t), t)=0,\quad \forall j\\
\text{a.e. } \sigma_k(t)(Q_k(t), u^0(t))=0,\quad \forall k.
\end{array}\right\} \tag{21}
$$

*This term may be not entirely appropriate; a better term is probably "differential maximum principle," but we follow the established convention.

Let us recall the definition of the important notion of stationarity for constrained extremum problems.

In some Banach space Z consider the problem

$$F_0(z)\to\min,\quad F_i(z)\leqslant 0,\quad i=1,\ \dots,\ \nu;\quad g(z)=0, \tag{22}$$

where g maps Z to some Banach space Y.

Definition 4. The point z_0 is stationary in problem (22) if there does not exist $\bar{z}$ such that $\forall i \geq 0$ $F_i'(z_0, \bar{z}) < 0$ and $\bar{z}$ is tangent at the point z_0 to the set $g(z) = 0$.

If z_0 is a local minimum point (in the topology induced by the norm), then z_0 is a stationary point. Stationarity is thus a necessary first-order condition of local maximum. As an independent object, the stationarity property has been systematically identified and analyzed in the work of Milyutin and Dubovitskii. Note that the point z_0 may be stationary both when the Lyusternik condition is satisfied ($g'(z_0)Z = Y$) and not.

It follows from the above that for problem B stationarity provides a necessary first-order condition of weak minimum.

THEOREM 1. The local maximum principle for the point (x^0, u^0) in problem B is equivalent to the following: either this point is stationary or the Lyusternik condition does not hold at this point. Thus,

Local Maximum Principle $\Leftrightarrow$ (stationarity OR "not Lyusternik").

This theorem is not the focus of our paper and in principle it is well known. We will therefore give only a brief sketch of the proof.

Of special interest to us is the implication $\Leftarrow$. We first assume that on the given trajectory

a) the Lyusternik condition holds;

b) $|\varphi'_{ix_0}(x_0^0, x_1^0)|+|\varphi'_{ix_1}(x_0^0, x_1^0)|\neq 0\quad \forall i$;

c) $|\Phi'_{sx}(x^0(t), t)|\geqslant \text{const}>0\ \ \forall s$.

This and assumption B2 show that all inequality constraints have nonempty feasible direction cones [9] and these cones, as well as the cone of tangential directions to the intersection of the equality constraints, are written in standard form as a linearization of constraints. In this case, the local maximum principle is easily proved by the Dubovitskii–Milyutin scheme [9]. By definition, stationarity implies that the constraint cones are nonintersecting. We can express this condition in dual form (Dubovitskii–Milyutin theorem): the sum of the elements in conjugate cones is zero (this equality is called the Euler equation). The form of the elements from conjugate cones is obtained in the usual way by multiplying linearized constraints by the corresponding Lagrange multipliers (see [9]).

Here we come to the main point of danger: *a priori*, m, h, β, and σ are elements of the space L_∞^*. However, assumption B2 on regularity of mixed constraints guarantees (see Lemma 1 below) – and this is indeed the strongest simplification of the regular problem compared with the nonregular one – that in fact m, h, β, σ are functions from L_1. Then the functionals h_j are *a priori* concentrated on each of the sets M_j^δ ($\delta > 0$),† but $h \in L_1$ and thus obviously h_j are concentrated on the sets M_j^0, i.e., they satisfy the complementary slackness conditions (21). We similarly obtain (21) for σ_k. We have thus established the local maximum principle.‡

If at least one of the conditions a), b), c) does not hold, i.e., some of the constraints are degenerate, then they are written (again by a standard technique) as a local maximum principle in which nonzero multipliers are attached only to the degenerate constraints (a so-called degenerate local maximum principle).

This completes the proof of the implication $\Leftarrow$.

Remark. Since $\dot{x} - f(x, u, t) \in L_1$, the Lagrange multiplier corresponding to equality (15) is *a priori* $\psi \in L_\infty$. But from Euler's equation (20) we easily obtain that almost everywhere

$$\psi(t)=\text{const}-\int_0^t \bar{H}_x' d\tau,$$

whence it follows that ψ is indeed a function of bounded variation.

†We say that the functional $h \in L_\infty^*$ is concentrated on the set $M \subset T$ if $\forall v \in L_\infty$ $\langle h, v\rangle = \langle h, \chi_M v\rangle$.

‡For a problem with nonregular mixed constraints, even the derivation of the local maximum principle is a highly nontrivial problem. It has been solved in [10-13].

We do not prove the converse implication $\Rightarrow$. We only note that this implication, like Theorem 1, is a particular case of the general theorem for an abstract problem of the form (22): the Euler equation at the point z_0 is equivalent to the condition that either z_0 is stationary or the Lyusternik condition does not hold at this point. And we have shown above that for problem B the local maximum principle is in fact the Euler equation.

3. LEMMA ON NONEXISTENCE OF SINGULARITIES

We will prove the previously promised lemma, which in our view is also of independent interest.

LEMMA 1 (on nonexistence of singular components). Suppose that the system of functions $\{A_i(t)\}$, $\{B_j(t)\}$ from $L_\infty{}^r$ is uniformly in t linearly positively independent with respect to the system of sets $\{M_j\}$.

Let m_i, $h_j \in L_\infty{}^*$, among which $h_j \geq 0$ and are concentrated on M_j, and

$$\sum_i m_i A_i + \sum_j h_j B_j = l \in L_1{}^r. \tag{23}$$

Then m_i, $h_j \in L_1$.

In the proof of Theorem 1, (23) is taken in the form of the Euler equation. More precisely, its component with respect to $\overline{u}$ is

$$\overline{H}_u = \psi f_u - m g_u - \beta p - h G_u - \sigma Q,$$

and $A_i = (g_{iu}{}', p)$, $B_j = (-G_{ju}{}', -Q_k)$.

Proof. Recall that any element $m \in L_\infty{}^*$ is representable in the form $m = m_a + m_c$, where

$$\forall v \in L_\infty \quad \langle m_a, v \rangle = \int s(t) v(t)\, dt, \quad s \in L_1$$

(the absolutely continuous component of the functional m) and m_c is the singular component. For the singular component there exists a sequence of nested measurable sets $\mathscr{E}_n \subset [t_0, t_1]$, mes $\mathscr{E}_n \to 0$, such that $\forall v \in L_\infty$

$$\langle m_c, v \rangle = \langle m_c, \chi_{\mathscr{E}_n} v \rangle \quad \forall n,$$

i.e., the functional m_c is concentrated on each of the sets $\mathscr{E}_n$. If the functional m is nonnegative, then m_a, m_c are also nonnegative; if m is concentrated on some set M, then m_a, m_c are concentrated on M and all $\mathscr{E}_n \subset M$ (Yosida–Hewitt theorem [20]; in this form, it was independently established also by Dubovitskii and Milyutin in [10]).

We have to show that under the conditions of the lemma the functionals m_i, h_j do not have singular components. Assume the contrary: we first assume that the functional h_1 has singular components. Since the vectors $A_i(t)$, $B_j(t)$ are ULPI with respect to the sets M_j, by Definition 2b it follows that there exists $\overline{u} \in L_\infty{}^r$ such that

$$\begin{aligned} &\forall i \ (A_i(t), \overline{u}(t)) = 0 \text{ a.e. on } T, \\ &\forall j \ (B_j(t), \overline{u}(t)) \geqslant 1 \text{ a.e. on } M_j. \end{aligned} \tag{24}$$

This and (23) give

$$\sum_j \langle h_j, (B_j, \overline{u}) \rangle = \int (l, \overline{u})\, dt. \tag{25}$$

Let $\|h_{1c}\| = \gamma > 0$ and assume that the functional h_{1c} is concentrated on a descending sequence of sets $\mathscr{E}_n$. Then for the sequence of functions $\overline{u}_n = \chi_{\mathscr{E}_n} \overline{u}$ we obtain by (24) and nonnegativity of the functionals h_j that the left-hand side of equality (25) is

$$\geqslant \langle h_{1c}, \chi_{\mathscr{E}_n} \rangle = \langle h_{1c}, 1 \rangle = \|h_{1c}\| = \gamma > 0,$$

and the right-hand side obviously tends to zero. A contradiction. Thus, h_j does not have singular components, i.e., all $h_j \in L_1$. Relationship (23) takes the form

$$\sum_j m_i A_i = \tilde{l} \in L_1{}^r. \tag{26}$$

Now assume that the functional m_1 has a singular component and let $\nu(t)$ be a function from L_∞ such that $\langle m_{1c}, \nu\rangle = 1$. Since the vectors $A_i(t)$ are ULPI, from Definition 2b it easily follows that there exists $\bar{u}_1 \in L_\infty{}^r$ such that

$$\text{a.e. } (A_1(t), \bar{u}_1(t)) = \nu(t), \text{ and } \forall i \geqslant 2 \quad (A_i(t), u_1(t)) = 0.$$

Assume that the functional m_{1c} is concentrated on a descending sequence of sets $\mathscr{E}_n$. Then, as before, for $\bar{u}_n = \chi_{\mathscr{E}_n} u$ the left-hand side of (26) tends to 1 and the right-hand side tends to zero. Again a contradiction. The functionals m_i thus do not have singular components, i.e., all $m_i \in L_1$. Q.E.D.

The condition $\lambda \neq 0$ in the local maximum principle can be replaced with the normalization condition

$$|a|+|c|+\|d\mu\|_{C^*}+\|h\|_1+\|\sigma\|_1>0, \tag{27}$$

i.e., the multipliers ψ, m, β are not included (note that in our case $\|d\mu\|_{C^*} = \mu(t_1 + 0) - \mu(t_0)$). Indeed, if the left-hand side of (27) is zero, then by the local maximum principle

$$\dot{\psi} = -\psi f_x + \sum_i m_i g_{ix}, \quad \psi(t_0) = 0, \tag{28}$$

$$\bar{H} = \psi f_u - \sum m_i g_{iu} - \beta p = 0. \tag{29}$$

Since the vectors g_{iu}, p are linearly independent (uniformly in t), m_i and β are linearly expressible in terms of ψ from (29). Substituting this result in (28), we obtain that ψ satisfies a linear homogeneous equation with zero initial condition, and thus $\psi \equiv 0$. But then also $m_i \equiv 0$ and $\beta \equiv 0$, i.e., in general $\lambda = 0$, a contradiction.

Now that we have derived the local maximum principle in problem B, we turn to the original problem A using the associated problems of the form B.

We again stress the important ideological aspect: to find the minimum conditions of the original problem, we examine a whole family of associated problems (see [9, 11, 14, 19, 27]). The main difficulty here is how to write the associated problem. On the one hand, it should be sufficiently smooth to allow efficient variation; on the other hand, the family of these problems should be as rich as possible.

4. THE FAMILY OF ASSOCIATED PROBLEMS

Let (x^0, u^0, w^0) be a triple on which the original problem A attains a Pontryagin minimum.

Take a natural number N and any functions $u_1{}^0, \ldots, u_N{}^0, w_1{}^0, \ldots, w_N{}^0$ from the spaces $L_\infty{}^{d(u)}$, $L_\infty{}^{d(w)}$ respectively such that fora all $k = 1, \ldots, N$ the triple $(x^0, u_k{}^0, w_k{}^0)$ satisfies the mixed and the nonfunctional constraints:

$$\left.\begin{array}{l} g(x^0, u_k{}^0, w_k{}^0, t) = 0, \\ G(x^0, u_k{}^0, w_k{}^0, t) \leqslant 0, \\ w_k{}^0(t) \in W(t). \end{array}\right\} \tag{30}$$

Let $u_0{}^0 = u^0$, $w_0{}^0 = w^0$, and consider a new problem C (the associated problem):

$$\dot{x} = \sum_{k=0}^{N} \alpha_k(t) f(x, u_k, w_k{}^0, t), \tag{31}$$

$$\Phi_s(x, t) \leqslant 0, \tag{32}$$

$$g(x, u_k, w_k{}^0, t) = 0, \tag{33}$$

$$G(x, u_k, w_k{}^0, t) \leqslant 0, \quad k = 0, 1, \ldots, N \tag{34}$$

$$\alpha_k(t) \geqslant 0, \tag{35}$$

$$\sum_{k=0}^{N} \alpha_k(t) = 1, \tag{36}$$

$$K(x_0, x_1) = 0, \tag{37}$$

$$J = \varphi_0(x_0, x_1) \to \min, \quad \varphi_i(x_0, x_1) \leqslant 0. \tag{38}$$

In this problem, the controls are $\vec{u} = (u_0, \dots, u_N) \in (L_\infty{}^r)^{N+1}$, $\vec{\alpha} = (\alpha_0, \dots, \alpha_N) \in L_\infty{}^{N+1}$. The tuple $\vec{w}^0 = (w_0{}^0, \dots, w_N{}^0) \in (L_\infty{}^{r'})^{N+1}$ is not varied.

Note that without w problem C is an extension of the original problem A, with new controls added; problem C with w, in general, is not an extension of problem A.

It is easy to see that problem C is a problem of type B and assumptions B1, B2 are satisfied. The mixed constraints (33)-(36) are regular because (33), (34) are regular by assumption A6, while the independent constraints (35), (36) are regular from obvious considerations.

Let us now state an important property of equality constraints in problem C.

THEOREM 2. Assume that the multitriple $(x, \vec{u}, \vec{\alpha})$ satisfies all the equality constraints of problem C. Also assume that

a) $\text{vraimin}_t\ \alpha_k(t) > 0\ \forall k = 0,1, \dots, N;$ (39)

b) the operator P defining the equality constraints (31), (33), (36), (37) satisfies the Lyusternik condition at the point $(x, \vec{u}, \vec{\alpha})$.

Then there exists a sequence of multitriples $(x^{(n)}, \vec{u}^{(n)}, \vec{\alpha}^{(n)})$ which also satisfy all the equality constraints and such that

$$\| x^{(n)} - x \|_\infty \to 0, \quad \| \vec{u}^{(n)} - \vec{u} \|_\infty \to 0, \quad (\vec{\alpha}^{(n)} - \vec{\alpha}) \xrightarrow{*-\text{weak}} 0,$$

where each function $\alpha_k{}^{(n)}(t)$ takes at most two values 0 and 1.

The proof is omitted here; it can be found in [26]. Note that [26] contains an error, which has been pointed out by Chukanov and subsequently corrected by the author. The correct proof is given in Chukanov's paper [38]. The simplest version of Theorem 2 was previously used in [9, Theorem 8.1].

The relationship between problems A and C is provided by Theorem 3 below, which relies on the following lemma.

LEMMA 2 [26]. Suppose that the measurable functions $\alpha_k(t)$, $k = 0, \dots, N$, are such that almost everywhere

$$\forall_k \quad \alpha_k(t) \geqslant 0, \quad \sum_k \alpha_k(t) = 1, \tag{40}$$

and

$$\alpha_0(t) \geqslant 1 - \delta, \quad \text{where } \delta > 0. \tag{41}$$

Assume that the sequence $\alpha_k{}^{(n)}(t)$ for all n also satisfies (40) and for all k we have $\alpha_k{}^{(n)} \to^{*\text{-weak}} \alpha_k$. Then for large n,

$$\sum_k \| \alpha_k^{(n)} - \alpha_k \|_1 < 6\delta.$$

Proof. Note that if $\alpha(t) \geq 0$, then $\|\alpha\|_1 = \int \alpha(t)dt$, i.e., for nonnegative functions the L_1-norm is a linear functional.

This result and the conditions of the lemma give

$$\begin{aligned} \| \alpha_0^{(n)} - \alpha_0 \|_1 &\leqslant \| \alpha_0^{(n)} - 1 \|_1 + \| 1 - \alpha_0 \|_1 = \int (1 - \alpha_0^{(n)})\, dt + \int (1 - \alpha_0)\, dt = \\ &= \int (1 - \alpha_0)\, dt + \int (\alpha_0 - \alpha_0^{(n)})\, dt + \int (1 - \alpha_0)\, dt < 3\delta \end{aligned}$$

for large n. Now, since $\alpha_0 + \Sigma_{k=1}{}^N \alpha_k = 1$, we have a.e. $\Sigma_{k=1}{}^N \alpha_k(t) < \delta$. Therefore, similarly to the above, we have

$$\begin{aligned} \sum_{k=1}^{N} \| \alpha_k^{(n)} - \alpha_k \|_1 &\leqslant \sum_k \| \alpha_k^{(n)} \|_1 + \sum_k \| \alpha_k \|_1 = \sum_k \int \alpha_k^{(n)} dt + \sum_k \int \alpha_k dt = \\ &= \sum_k \int (\alpha_k^{(n)} - \alpha_k)\, dt + 2 \sum_k \int \alpha_k dt < 3\delta \end{aligned}$$

for large n. Q.E.D.

Remark. This lemma admits a more general statement. Specifically, let $A(t)$ be an arbitrary convex compactum in R^N that measurably depends on t and is included in a ball of radius $\rho(t) \in L_1$. Let ex $A(t)$ be the set of extreme points of this compactum. Let almost everywhere $\alpha(t) \in A(t)$ and $\text{dist}(\alpha(t), \text{ex } A(t)) \leq \xi(t)$, where $\int \xi(t)dt = \delta$; a.e. $\alpha^{(n)}(t) \in A(t)$

and $\alpha^{(n)} \to^{w} \alpha$ (weakly in L_1). Then $\forall \varepsilon > 0$ and large n, $\|\alpha^{(n)} - \alpha\|_1 < f(\varepsilon + \delta) + \delta$, where $f(s) \to 0$ as $s \to 0$. The proof is left to the reader. Note that if $\rho \notin L_1$, the lemma does not hold.

THEOREM 3. Suppose that a Pontryagin minimum in the original problem A is attained on the triple $\tau^0 = (x^0, u_0{}^0, w_0{}^0)$ and that in problem C the Lyusternik condition is satisfied for the multitriple

$$\rho^0 = (x^0, \vec{u}^0, \vec{\alpha}_0), \quad \text{where} \quad \vec{\alpha}^0 = (1, 0, \ldots, 0), \tag{42}$$

Then ρ^0 is stationary in problem C.

Proof. By contradiction. Assume that the multitriple ρ^0 is not stationary. Then for an arbitrary $\varepsilon > 0$ at a distance ε from this multitriple (in the L_∞ metric) there exists a multitriple $\rho = (x, \vec{u}, \vec{\alpha})$ which satisfies all the equalities of problem C and is included inside all the inequalities. In particular, $J(\rho) < J(\rho^0) = J(\tau^0)$ and (39) holds.

Since the Lyusternik condition is satisfied for ρ^0 and the operator P defining the equalities of problem C is uniformly (Fréchet) differentiable, then the Lyusternik condition is also satisfied for the multitriple ρ close to ρ^0. Also note that $\vec{\alpha}$ satisfies (41).

By Theorem 2 there exists a multitriple $\rho' = (x', \vec{u}', \vec{\alpha}')$ that satisfies all the constraints of problem C and is closed (in the relevant sense) to ρ. We may take $\|x' - x\|_\infty < \varepsilon$, $\|\vec{u}' - \vec{u}\|_\infty < \varepsilon$, and by Lemma 2 $\|\vec{\alpha}' - \vec{\alpha}\|_1 < 6\varepsilon$. Moreover, we may assume that ρ', like ρ, satisfies as strict inequalities all the inequalities of problem C that do not contain $\vec{\alpha}$; in particular, $J(\rho') < J(\tau^0)$ as before.

But (!) $\alpha_k'(t) = 0$ or 1, i.e., $\alpha_k'(t) = \chi_{\mathscr{E}_k}(t)$, where the measurable sets $\mathscr{E}_k$ form a partition of the interval $[t_0, t_1]$, and thus setting

$$u'(t) = \sum_k \alpha_k'(t) u_k'(t), \quad w'(t) = \sum_k \alpha_k'(t) w_k{}^0(t),$$

we obtain an "ordinary" triple $\tau' = (x', u', w')$ (with the same x') that satisfies all the constraints of problem A and $J(\tau') = J(\rho')$. Thus,

$$J(\tau') < J(\tau^0). \tag{43}$$

Here

$$\|u' - u^0\|_1 \leqslant \sum_k \int |\alpha_k' u_k' - \alpha_k{}^0 u_k{}^0|\, dt \leqslant \sum_k \int |\alpha_k' u_k' - \alpha_k' u_k{}^0| + |\alpha_k' u_k{}^0 - \alpha_k{}^0 u_k{}^0|\, dt \leqslant \varepsilon + \|\vec{u}^0\|_\infty \cdot \varepsilon 6,$$

and similarly $\|w' - w^0\|_1 \le \|\vec{w}^0\|_\infty \cdot 6\varepsilon$. Hence, noting that ε is arbitrary and using (43), we see that there is no Pontryagin minimum at the point $\tau^0 = (x^0, u^0, w^0)$ in problem A. A contradiction. Q.E.D.

COROLLARY. By Theorem 3 and Theorem 1, if $\tau^0 = (x_0, u^0, w^0)$ is a Pontryagin minimum point in problem A, then in any associated problem C the multitriple (42) satisfies the local maximum principle irrespective of the Lyusternik condition.

5. FINITE-VALUED MAXIMUM PRINCIPLE

We now write out the local maximum principle for the multitriple (42) (using the results of Secs. 2, 3).

There exists a tuple

$$\lambda = (a, c, \psi, \mu, m_k, h_k, \beta, \sigma_k),$$

where a, c, ψ, μ are from the same spaces as before, $a \geqslant 0$, $d\mu \geqslant 0$, $m_k \in L_1^{d(g)}$, $h_k \in L_1^{d(G)}$, β, $\sigma_k \in L_1$, $h_k \geqslant 0$, $\sigma_k \geqslant 0$,

$$|a| + |c| + \|d\mu\| + \sum \|h_k\|_1 + \sum \|\sigma_k\|_1 > 0; \tag{44}$$

such that the functions $l = (a, \varphi) + (c, K)$,

$$\begin{aligned} \bar{H} = {} & \sum \alpha_k (\psi, f(x, u_k, w_k{}^0)) - (\dot{\mu}, \Phi(x, t)) - \\ & - \sum (m_k, g(x, u_k, w_k{}^0)) - \sum (h_k, G(x, u_k, w_k{}^0)) - \beta(t) \sum \alpha_k(t) + \\ & + \sum \sigma_k(t) \alpha_k(t), \end{aligned}$$

on the trajectory (42) satisfy the equalities

$$-\dot{\psi}=\bar{H}_x=\sum \alpha_k^0\psi f_x(x^0,u_k^0,w_k^0)-\dot{\mu}\Phi_x(x^0,t)-\sum m_k g_x(x^0,u_k^0,w_k^0)-\sum h_k G_x(x^0,\mu_k^0,w_k^0), \tag{45}$$

$$\psi(t_0)=l'_{x_0},\quad \psi(t_1)=-l'_{x_1}, \tag{46}$$

$\bar{H}_{u_k}=0$, $\bar{H}_{\alpha_k}=0$, and also the complementary slackness conditions.

Since $\bar{\alpha}^0=(1,0,\dots,0)$, we have

$$\begin{aligned}&\bar{H}_{u_0}=(\psi f_u-m_0g_u-h_0G_u)(x^0,u_0^0,w_0^0)=0,\\ &\bar{H}_{u_k}=(-m_kg_u-h_kG_u)(x^0,u_k^0,w_k^0)=0,\quad k=1,\dots,N.\end{aligned} \tag{47}$$

From the last equality by linearly positive independence of g_u, G_u, we obtain

$$m_k=0,\quad h_k=0,\quad \forall k=1,\dots,N.$$

And then (45) takes the form

$$-\dot{\psi}=(\psi f_x-\dot{\mu}\Phi_x-m_0g_x-h_0G_x)(x^0,u_0^0,w_0^0). \tag{48}$$

From the equalities

$$\bar{H}_{\alpha_k}=(\psi,f(x^0,u_k^0,w_k^0))-\beta+\sigma_k=0,\quad \forall k\geqslant 0 \tag{49}$$

and the complementary slackness conditions $\sigma_k(t)\alpha_k^0(t)=0$ $\forall k\geq 0$ we obtain that $\sigma^0(t)\equiv 0$, and therefore the function

$$H(x,u,w,t)=(\psi,f(x,u,w,t))$$

satisfies almost everywhere the inequalities

$$H(x^0,u_k^0,w_k^0,t)\leqslant H(x^0,u_0^0,w_0^0,t)\quad \forall k=1,\dots,N. \tag{50}$$

Moreover, the complementary slackness conditions are satisfied:

$$\left.\begin{aligned}&a_i\varphi_i(x_0^0,x_1^0)=0,\quad \forall i\geqslant 1,\\ &d\mu_s\cdot\Phi_s(x^0(t),t)\equiv 0,\quad \forall s,\\ &h_{0j}(t)G_j(x^0,u_0^0,w_0^0,t)\equiv 0,\quad \forall j.\end{aligned}\right\} \tag{51}$$

Now note that (44) can be replaced with the normalization

$$|a|+|c|+\|d\mu\|+\|h_0\|_1=1. \tag{52}$$

The remaining multipliers ψ, m_0, β, σ_k are linearly expressible in terms of these multipliers. (Recall that for $k\geq 1$ we have $m_k=0$, $h_k=0$.)

Indeed, from (37) by uniform in t nondegeneracy of g_u we obtain a linear expression of m_0 in terms of ψ, h_0, and then from (48), (46) we obtain $\psi[a,c,\mu,h_0]$ and then m_0. From (49), noting that $\sigma_0=0$, we obtain $\beta=(\psi,f(x^0,u_0^0,w_0^0,t))$ and then find σ_k.

Alternatively, if a, c, μ, h_0 are zero, then from (37) we express m_0 in terms of ψ and substituting in (48) we obtain a homogeneous equation for ψ with zero initial conditions (46). Thus, $\psi=0$ and $m_0=0$. Then from (49) for $k=0$ we obtain $\beta=0$, and so $\sigma_k=0$ $\forall k\geq 1$. Thus, the entire tuple $\lambda=0$, which contradicts (44).

6. GLOBAL MAXIMUM PRINCIPLE

We have thus shown that if $\tau^0=(x^0,u_0^0,w_0^0)$ is a Pontryagin minimum point in problem A, then for any tuple

$$z=(u_1^0,\dots,u_N^0;\ w_1^0,\dots,w_N^0),$$

satisfying the constraints (30) there exists a tuple

$$\lambda=(a,\ c,\ \psi,\ d\mu_s,\ m=m_0,\ h=h_0),$$

where $a\in R_+^{d(\varphi)+1}$, $c\in R^{d(k)}$, $\psi\in L_\infty^{d(x)}$, $d\mu_s(t)\geqslant 0$ is the Radon measure, i.e., $d\mu_s \in C^*$,

$$m\in L_1^{d(g)}\subset(L_\infty^{d(g)})^*,\quad h\in L_1^{d(G)}\subset(L_\infty^{d(G)})^*,\quad h(t)\geqslant 0,$$

which satisfies the finite-valued maximum principle (46), (47), (48), (50), (51) and the normalization condition (52).

Although we know that $m^{(i)}$, $h^{(j)} \in L_1$, it is convenient to take $m^{(i)}$, $h^{(j)} \in L_\infty{}^*$ (in this case it follows from (47) and Lemma 1 that they are elements of L_1). The normalization condition is therefore written in the form

$$|a|+|c|+\sum_s\int d\mu_s(t)+\sum_j\langle h^{(j)},\ 1\rangle=1. \tag{53}$$

Here the function $\mathbf{1}(t) \equiv 1$, $h^{(j)}$ is the j-th component of the vector functional $h = h_0$.

The set of all tuples λ with these properties is denoted by

$$\Lambda(z)=\Lambda(u_1^0,\ \ldots,\ u_N^0;\ w_1^0,\ \ldots,\ w_N^0),$$

and the superscript 0 is henceforth omitted. This set is contained in the space

$$Y^*=R^{d(\varphi)+1}\times R^{d(K)}\times L_\infty^{d(x)}\times(C^*)^{d(\Phi)}\times(L_\infty^*)^{d(g)}\times(L_\infty^*)^{d(G)},$$

which is the conjugate of the corresponding Banach space Y. Since conditions (46), (47), (48), (50), (51) are linear in λ and the normalization (53) is linear in both ∞-dimensional components ($d\mu$ and h), we conclude that $\Lambda(z)$ is *-weak closed.

Since a, c, $d\mu$, h are bounded by equality (53) and the other multipliers are linearly expressible in terms of these four multipliers (clearly in a bounded form), the set $\Lambda(z)$ is bounded.

Hence by Alaoglu's theorem we obtain that $\Lambda(z)$ is a *-weak compactum, and we know that it is nonempty.

Thus, for any tuple $z = (u_1, \ldots, u_N; w_1, \ldots, w_N)$ that satisfies (30) there is a nonempty compactum $\Lambda(z)$ in the space Y^* in which every point satisfies the conditions (46), (47), (48), (51), (53) regardless of z and also the maximum condition (50) for z.

Clearly, if z'' is larger than z' (in the natural sense), then $\Lambda(z'') \subset \Lambda(z')$, and therefore $\forall z'$, z'' the intersection $\Lambda(z') \cap \Lambda(z'')$ is nonempty, because it contains the nonempty set $\Lambda(z' \cup z'')$. Thus, $\{\Lambda(z)\}$ is a centered system of compacta and it accordingly has a nonempty intersection.

Take any $\lambda = (a,\ c,\ \psi,\ d\mu,\ m,\ h) \in \cap_z \Lambda(z)$. It satisfies the following global maximum principle.

Global Maximum Principle

a) the normalization condition (52),

b) the complementary slackness conditions (51),

c) the conjugate equation

$$-\dot{\psi}=\overline{H}_x=\psi f_x-\dot{\mu}\Phi_x-mg_x-hG_x,$$

where

$$\overline{H}(x,\ u,\ w,\ t)=(\psi,\ f)-(\dot{\mu},\ \Phi)-(m,\ g)-(h,\ G) \tag{48}$$

is the temporal (i.e., time-dependent) Lagrange function,

d) the transversality conditions

$$\psi(t_0)=l'_{x_0},\qquad \psi(t_1)=-l'_{x_1}, \tag{46}$$

where $l = (a,\ \varphi(x_0,\ x_1)) + (c,\ K(x_0,\ x_1))$ is the endpoint Lagrange function,

e) the Euler equation in u,

$$\overline{H}_u=\psi f_u-mg_u-hG_u=0, \tag{47}$$

f) the maximum condition: any measurable $u(t)$, $w(t)$ for which almost everywhere

$$\left.\begin{array}{l} g(x^0(t),\ u(t),\ w(t),\ t)=0, \\ G(x^0(t),\ u(t),\ w(t),\ t)\leqslant 0, \\ w(t)\in W(t), \end{array}\right\} \tag{54}$$

satisfy a.e. the inequality

$$H(x^0(t),\ u(t),\ w(t),\ t)\leqslant H(x^0(t),\ u^0(t),\ w^0(t),\ t), \tag{55}$$

where $H = (\psi, f)$ is the Pontryagin function.

If the multivalued mapping

$$V(t)=\{(u,\ w)\in R^{r+r'}\,|\,g(x^0(t),\ u,\ w,\ t)=0,$$
$$G(x^0(t),\ u,\ w,\ t)\leqslant 0,\ w\in W(t)\}$$

is measurable (for this it suffices to have $W(t)$ measurable), then standard arguments of the theory of measurable multivalued mappings suggest that condition f) is replaced with the condition

f′) for almost all t,

$$\max_{(u,w)\in V(t)} H(x^0(t),\ u,\ w,\ t)=H(x^0(t),\ u^0(t),\ w^0(t),\ t),$$

i.e., the maximum condition for H in the traditional form.

That is all. We have derived the necessary conditions of a Pontryagin minimum for the original problem A.

Remark. The standard form of the maximum principle usually contains an additional condition

$$\dot{H}=\frac{\partial H}{\partial t}, \tag{56}$$

which is missing from our formulation. Sliding variations do not derive this condition directly. It is obtained with the aid of so-called ν-variations [9-14, 27] (which involve a special time substitution), but this requires smoothness of the problem in t (recall that in problem A all functions, except Φ, are only measurable in t).

Thus, for a problem smooth in t, sliding variations produce a result which is only slightly weaker than that obtained with ν-variations. Perhaps condition (56) can be derived from the maximum principle (a)-(f)? The answer is no: Milyutin supplied an example in which a t-smooth problem satisfies the maximum principle (a)-(f) but violates condition (56).

In problems with nonregular mixed constraints, the difference between sliding variations and ν-variations is even more substantial (see [14]).

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