Nonnegativity of a degenerate quadratic form and a two-dimensional inequality of Hardy type

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A quadratic form with Legendre condition degenerate at a single point:

$$J = \int_0^1 \big(t^2(u,u) - 2bt(Px,u) + (Dx,x)\big)\,dt, \tag{1}$$

$$\dot{x} = u, \qquad x(1) = 0. \tag{2}$$

Here, x and u are two-dimensional, P is the matrix of rotation by 90°, $b \in \mathbf{R}$ – a parameter, D – a constant symmetric matrix, $u(t) \in L_\infty[0,1]$.

Functional (1, 2) is self-similar: its nonnegativity does not depend on the interval $[0,T]$; so we set $T = 1$.

The problem: describe parameters b and D for which $J \geq 0$ on the above set of functions.

We cannot directly apply the classical Jacobi condition.

Lemma 1. *$J(x) \geq 0$ for all Lipschitzian $x(t)$ with $x(1) = 0$ is equivalent to $J(x) \geq 0$ for all Lipschitzian $x(t)$ with $x(0) = x(1) = 0$.*

This lemma makes it possible to apply the Jacobi conditions, since we can assume that $x(0) = 0$, and then we can move the left endpoint of this interval. For $\theta > 0$, the strengthened Legendre condition holds on each interval $[\theta, 1]$, and, therefore, we can seek out a point t_* conjugate to $t = 1$.

If there is no such point on (0,1), then for any $\theta > 0$ the functional $J > 0$ on $[\theta, 1]$, and then $J \geq 0$ on $[0, 1]$.

But we proceed in another way, by extracting a complete square.

Functional (1, 2) can be rewritten in the following two interesting forms. Set $t = e^{-\tau}$, $tu = -w(\tau)$, then $dt = -t d\tau$, and

$$J = \int_0^\infty e^{-\tau} \left[w^2 + 2b(Px, w) + (Dx, x) \right] d\tau, \tag{3}$$

$$\frac{dx}{d\tau} = w, \qquad x(0) = 0.$$

Such functionals are typical for mathematical economics models.

Further, set $e^{-\tau/2} x = z$ and $e^{-\tau/2} w = v$, then

$$\frac{dz}{d\tau} = -\frac{1}{2} z + v, \qquad z(0) = 0, \tag{4}$$

and the functional has constant coefficients:

$$J = \int_0^\infty \left[v^2 + 2b(Pz, v) + (Dz, z) \right] d\tau. \tag{5}$$

Eq. (4) can be simplified by setting $u = -\frac{1}{2} z + v$. Changing the notation τ, z to the usual t, x, we obtain

$$\dot{x} = u, \qquad x(0) = 0, \tag{6}$$

$$J = \int\limits_0^\infty \left[u^2 + 2b(Px, u) + (Qx, x)\right]\,dt, \tag{7}$$

where $Q = D + \frac{1}{4}E$.

Our functional will be studied precisely in this form.

Lemma 2. *If* $J \geq 0$, *then* $Q \geq 0$ *(a condition of Legendre type).*

(This condition does not hold for a finite interval $[0, T]$!)

3. Take an arbitrary symmetric matrix S and add the expression

$$\frac{d}{dt}\,(Sx, x) = 2(Sx, u)$$

under the integral sign in order to extract a complete square of terms containing u. We obtain

$$J = \int_0^\infty \big([u + (S + bP)x]^2 + (Mx, x)\big)\,dt, \tag{8}$$

where

$$(Mx, x) = (Qx, x) - (Sx + bPx)^2,$$

i.e.,

$$M = Q - S^2 + b(PS - SP) - b^2 E. \tag{9}$$

If $M \geq 0$, then, obviously, $J \geq 0$ for all compactly supported functions satisfying (6).

We ask the question: for which b and Q can one obtain $M \geq 0$ by choosing an appropriate matrix S ?

Without loss of generality, we assume that matrix Q is diagonal, i.e.,

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix},$$

where, according to Lemma 2, $q_1 \geq 0$ and $q_2 \geq 0$.

Let us seek S in the form

$$S = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

where c is an unknown parameter. (Later we will see that such S are sufficient.) Then, by simple calculations, we get

$$M = \begin{pmatrix} q_1 - (b+c)^2 & 0 \\ 0 & q_2 - (b-c)^2 \end{pmatrix}, \tag{10}$$

and the question reduces to the following: for which b, q_1, q_2 there exists c such that

$$q_1 - (b+c)^2 \geq 0, \qquad q_2 - (b-c)^2 \geq 0 \ ?$$

The latter is equivalent to inequalities

$$\sqrt{q_1} \geq |b+c|, \qquad \sqrt{q_2} \geq |b-c|.$$

Obviously, such c can be found if and only if

$$\frac{1}{2}\left(\sqrt{q_1} + \sqrt{q_2}\right) \geq |b|. \tag{11}$$

Thus, we established that if (11) is fulfilled, then there exists a symmetric matrix S such that the corresponding $M \geq 0$, and, therefore, $J \geq 0$.

What happens if (11) is not fulfilled? There is no matrix S of the required form (with zero diagonal) but does this mean that the property $J \geq 0$ is violated? We will show that it is really so, and this is a key point of our approach.

Without loss of generality, assume that $b \geq 0$. Let

$$\frac{1}{2}\left(\sqrt{q_1} + \sqrt{q_2}\right) < b.$$

In this case there obviously exists c such that

$$\sqrt{q_1} < b + c, \qquad \sqrt{q_2} < b - c \tag{12}$$

(for example, $c = \frac{1}{2}\,(\sqrt{q_1} - \sqrt{q_2})$), i.e.,

$$q_1^2 < (b+c)^2, \qquad q_2^2 < (b-c)^2;$$

hence, both entries of matrix M are negative: $M < 0$. Then, for a certain $\delta > 0$ we have $(Mx, x) \le -\delta|x|^2 \quad \forall\, x$, i.e., the second term in (8) is negative-definite.

Now, we try to find a pair (x, u) for which the first term in (8) is zero: $u + (S + bP)x = 0$, i.e.,

$$\dot{x} = -(S + bP)x. \tag{13}$$

Lemma 3. *If* (12) *holds, then Eq. (13) has a periodic solution of the form*

$$x(t) = f \sin \omega t + h \cos \omega t$$

with $\omega \neq 0$ *and linearly independent vectors* $f,\ h \in \mathbf{R}^2$.

Proof. It suffices to show that matrix $R = (S + bP)$ has a purely imaginary eigenvalue $\lambda = i\omega \neq 0$. Since

$$R = \begin{pmatrix} 0 & c - b \\ c + b & 0 \end{pmatrix},$$

the characteristic equation is $\lambda^2 + (b^2 - c^2) = 0$. If $b^2 - c^2 > 0$ (i.e., if $|c| < b$), then this equation has two purely imaginary roots

$$\lambda = \pm i\,\omega, \qquad \omega = \sqrt{b^2 - c^2} > 0.$$

Check that precisely this case is realized. From (12) we get $b + c > 0$, $b - c > 0$, whence $b^2 - c^2 > 0$. Lemma is proved. $\square$

Since the vectors f and h are linearly independent, $x(t) = f\sin\omega t + h\cos\omega t$ describes an ellipse in $\mathbf{R}^2$, and we can assume that $|x| \geq 1$ on it. Now, consider $x(t)$ on a large time interval $[1, T]$. The first term in (8) is equal to zero by definition, and since $|x| \geq 1$, we have $(Mx, x) \leq -\delta$; therefore, the integral over $[1, T]$ is a negative quantity $\leq -\delta(T-1)$ of order T.

On the intervals [0,1] and $[T, T+1]$ we reduce x to zero end-values at the points 0 and $T+1$; since $x(t)$ is bounded, the integrals over these intervals make only a finite contribution to the functional. Therefore, on the entire interval $[0, T+1]$, for large T we obtain a function $\hat{x}(t)$ for which

$$J(\hat{x}) \ \leq \ -\delta T + \mathrm{const} < 0\ .$$

Remark 1. The key point here is the property that the found cyclic solution of Eq.(13) can be "rolled up" for an arbitrarily long time, thereby accumulating an arbitrarily large negative integral of (Mx, x) and preserving the bounded value of x. The first term in (8) remains equal to zero all the time by virtue of (13).

Thus, if inequality (11) does not hold, there exists a compactly supported function $\hat{x}(t)$ for which $J(\hat{x}) < 0$. Therefore, we established the following property.

Theorem 1. *Functional* (7) *is nonnegative on all compactly supported functions satisfying Eq.(6) if and only if the eigenvalues of matrix Q are nonnegative and satisfy* (11).

Remark 2. Squaring (11), we obtain an equivalent inequality $(q_1 + q_2) + 2\sqrt{q_1 q_2} \geq 4b^2$, which can be written without reducing the matrix Q to the diagonal form:

$$\mathrm{Tr}\, Q + 2\sqrt{\det Q} \ \geq \ 4b^2. \tag{14}$$

4. Particular cases. Examine the obtained conditions for functionals (1) and (3). Since $Q = D + \frac{1}{4}E$, inequality $Q \geq 0$ means $D + \frac{1}{4}E \geq 0$, and (11) means

$$\frac{1}{2}\left(\sqrt{d_1 + \frac{1}{4}} + \sqrt{d_2 + \frac{1}{4}}\right) \geq |b|. \tag{15}$$

(b) Consider the case $b = 0$ and $D = -\frac{1}{4}E$ (i.e., $Q = 0$). Then functional (3) satisfies the inequality

$$J = \int_0^\infty e^{-t}\left(u^2 - \frac{1}{4}x^2\right)\, dt \geq 0 \tag{16}$$

on all compactly supported $x(t)$ such that

$$\dot{x} = u, \qquad x(0) = 0. \tag{17}$$

In other words, for these functions,

$$\int_0^\infty e^{-t}x^2\, dt \;\leq\; 4\int_0^\infty e^{-t}u^2\, dt. \tag{18}$$

Introduce the Hilbert space $H = L_2[0, \infty)$ with weight e^{-t}. By (18), the integral operator $u \mapsto x$ given by formula (17) is a linear bounded operator $H \to H$, and its norm does not exceed $\sqrt{4} = 2$. Actually, its norm is equal to 2, since the constant 4 in inequality (18) is sharp.

(c) For functional (5), this property means that for any compactly supported function z satisfying (4), we have

$$J = \int_0^\infty \left(v^2 - \frac{1}{4}z^2\right)\, dt \geq 0,$$

i.e.,

$$\int_0^\infty z^2\, dt \;\leq\; 4\int_0^\infty v^2\, dt. \tag{19}$$

This implies that for any function $v \in L_2[0,\infty)$, the function $z(t)$ satisfying the equation

$$\dot z = -\frac{1}{2}\,z + v, \qquad z(0) = 0, \tag{20}$$

also belongs to $L_2[0,\infty)$, and, moreover, the norm of operator $v \mapsto z$ does not exceed (and actually equals) 2.

The same property holds for the equation

$$\dot z = -kz + v, \qquad z(0) = 0 \tag{21}$$

for any $k > 0$. (It reduces to (20) by simple scaling.)

Lemma 4. *For any $v \in L_2[0,\infty)$, the solution $z(t)$ of Eq. (21) also belongs to $L_2[0,\infty)$, and the norm of operator $v \mapsto z$ is equal to $1/k$.*

Remark 3. The above operator, although being integral and, therefore, completely continuous in the space $L_2[0,T]$ for any finite T, is not completely continuous in the space $L_2[0,\infty)$. Moreover, it has a purely continuous spectrum $=$ the disk in the complex plane, whose diameter is the segment $[0,1/k]$ of real axis.

(d) For functional (1) with $D = -\frac{1}{4}E$ and $b = 0$, we obtain the inequality

$$J = \int_0^1 \left(t^2u^2 - \frac{1}{4}x^2 \right) dt \ge 0,$$

$$\text{i.e.,} \qquad \int_0^1 x^2\, dt \;\le\; 4\int_0^1 t^2u^2\, dt \tag{22}$$

for all x, u satisfying Eq. (2) and vanishing in a certain neighborhood of $t=0$. Then the same inequality is also true for all $u(t)$ for which the integral on the right-hand side of (22) converges (i.e., for all $u(t)$ from the space $L_2[0,1]$ with weight t^2); here the integral on the left-hand side also converges by virtue of (22).

Inequality (22) and the related inequalities (19) and (18) are the well-known Hardy inequality [1, Sec. 9.8]; therefore, the inequality $J \geq 0$, under conditions (11) or (15), can be treated as its two-dimensional generalization.

For example, taking $Q = aE$ with arbitrary $a > 0$ in functional (3), we have $D = (a - 1/4)E$ and, according to (15), $|b| \leq a$, so, setting $b = \pm a$, we get

$$J = \int\limits_0^\infty e^{-t}\left[u^2 \pm 2a(Px,u) + (a - 1/4)x^2\right]\,dt \;\geq\; 0$$

under the conditions $\dot{x} = u, \qquad x(0) = 0.$

This inequality can be written as

$$\int\limits_0^\infty e^{-t}\left(\frac{1}{4} - a\right)|x|^2\,dt \;\leq\; \int\limits_0^\infty e^{-t}\left(u^2 \pm 2a(Px,u)\right)dt. \tag{23}$$

For functional (1) we similarly obtain

$$\int_0^1 \left(\frac{1}{4} - a\right)|x|^2\,dt \;\leq\; \int_0^1 \left(t^2\,|u|^2 \pm 2at(Px,u)\right)dt \tag{24}$$

under the conditions $\dot{x} = u, \qquad x(1) = 0.$

The two last inequalities are equivalent. Let us consider one of them, say (23). It can be rewritten as

$$\int\limits_0^\infty e^{-t}\left(\frac{1}{4} - a\right)|x|^2\,dt + 2a\left|\int\limits_0^\infty e^{-t}\,(Px,u)\right|dt \;\leq\; \int\limits_0^\infty e^{-t}\,|u|^2\,dt \tag{25}$$

for any compactly supported $x(t)$ with $\dot{x} = u,\;\; x(0) = 0.$

Note that (25) would easily follow from the Hardy case $(a = 0)$ if we had the inequality

$$-\int_0^\infty e^{-t}\,|x|^2\,dt \;+2\left|\int_0^\infty e^{-t}\,(Px,u)\right|dt \;\le\; 0,$$

i.e.,

$$2\left|\int_0^\infty e^{-t}\,(Px,u)\right|dt \;\le\; \int_0^\infty e^{-t}\,|x|^2\,dt, \tag{26}$$

but the last one is not true, because we can roll up $x(t)$ along any closed cycle with arbitrary large speed, thus obtaining a bounded quantity at the right hand side and an infinitely growing quantity at the left hand side of (26). So, one cannot obtain (23) just by adding (26) to the Hardy inequality.

5. We also call attention to the following interesting property.

Lemma 5. *Let inequality* (11) *or equivalent inequality* (14) *hold with the equality sign for functional* (7). *Then* $J(x) > 0 \quad \forall\, x \ne 0$.

Indeed, if (11) turns into equality, then the matrix M vanishes. Therefore, the functional J reduces to the first term in (8), and if $J(x) = 0$, then $u + (S + bP)x = 0$, i.e., $\dot{x} = -(S + bP)x$. Since $x(0) = 0$, we obtain $x \equiv u \equiv 0$. □

Thus, if (11) turns into equality, then, on the one hand, we have $J(x) > 0$; on the other hand, one cannot decrease any q_i or increase $|b|$, since inequality (11) would be violated, and by the same Theorem 1, we would obtain $J(x) < 0$ for some x.

Lemma 5 says that the constant 4 in inequalities (18), (19), and (22) is sharp but is not attained. That is, on the one hand, it cannot be decreased, and, on the other hand, for any nonzero function $x(t)$ these inequalities are strict, i.e., they are fulfilled with a certain $C(x) < 4$. This is an interesting peculiarity of inequalities between functions and their derivatives.

All these estimates can be also obtained either by the theory of conjugate points, or by the so-called frequency criterion [3, 4, 6]; this is done in [8].

One more corollary of Lemma 5 is the possibility of constructing examples of optimal control problems on $[0, \infty)$ which have no solution.

Example 1. The problem: $J(x, u) \to \min$, where J is given by (6), (7) with coefficients satisfying (11) in the form of the equality (for example, $b = q_1 = q_2 = 0$), under the constraint

$$\int_0^\infty |x|^2\, dt = 1. \tag{28}$$

Show that $\inf J = 0$ (and then, by Lemma 5, it is not attained).

Indeed, if $\inf J = a > 0$, then, by virtue of the homogeneity, for any $x \in L_2\,[0, \infty)$ we have $J(x) \geq a \int |x|^2\, dt$, i.e.,

$$\tilde{J}(x) = J(x) - a \int_0^\infty |x|^2\, dt \geq 0. \tag{29}$$

But $\tilde{J}$ has the matrix $\tilde{Q} = Q - aE$, that violates inequality (11) (because it is fulfilled as an equality for Q); therefore, by Theorem 1, inequality (29) cannot hold for all x, a contradiction.

Note that this problem is convex in u. One can object that it is not compact in u. Then, consider the following

Example 1'. The same problem with additional constraint $|u| \leq 1$. One can show that still $\inf J = 0$.

In the particular case where $b = q_1 = q_2 = 0$ (actually, this is a one-dimensional case), we obtain that the problem

$$J = \int_0^\infty u^2\,dt \to \min, \qquad \dot{x} = u, \quad x(0) = 0,$$

$$\int_0^\infty x^2\,dt = 1,$$

and also this problem with additional constraint $|u| \le 1$, has no solution. This phenomenon does not happen for a finite time interval $[0, T]$.

Similar considerations show that there is no solution in the following example of "economical" type.

Example 2.

$$J = \int_0^\infty e^{-\tau}\left(w^2 + 2b(Px, w) + (Dx, x)\right)\,d\tau \to \min,$$

$$\frac{dx}{d\tau} = w, \qquad x(0) = 0,$$

$$\int_0^\infty e^{-\tau}\,(x, x)\,d\tau \;= 1,$$

where b and D satisfy condition (15) in the form of equality (e.g., $b = 0$ and $d_1 = d_2 = -1/4$).

Here, as before, $\inf J = 0$, but it is not attained.

Conclusions. We studied the simplest nontrivial case of a quadratic functional with the degenerate Legendre condition by transforming it into a functional with "good" coefficients but defined on the semiaxis $[0, \infty)$. On the semiaxis (in general, on the spaces of infinite measure), the integral functionals qualitatively differ in their properties from the integral functionals on finite intervals (i.e., on the spaces of finite measure); they still have a singularity. This is related to the fact that the integral operators on the spaces of infinite measure are not, in general, completely continuous.

We have succeeded in obtaining exact formulas for the nonnegativity of the examined functional only in the two-dimensional case.

Even for the three-dimensional case, this question remains open.

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