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Second Order Necessary and Sufficient Conditions of a Pontryagin Minimum for Singular Boundary Extremals

We consider the following optimal control problem (Problem A):

$$J = \varphi_0(p) \to \min, \qquad \varphi_i(p) \le 0, \quad i = 1,\dots,\nu, \qquad g(p) = 0, \tag{1}$$

$$\dot{x} = f(x,t) + F(x)u \;=\; f(x,t) \;+\; \sum_{i=1}^{k} u_i\, f_i(x), \qquad u(t) \in U. \tag{2}$$

Here $p = (x_0, x_T),\ x_0 = x(t_0),\ x_T = x(T)$, the time interval $[t_0, T]$ is fixed; the variables u, x, g are multidimensional, $f_i(x)$ are the columns of the matrix $F(x)$. The set U is polyhedral, closed and solid.

Let us examine a trajectory $w^0(t) = (x^0(t), u^0(t))$, which satisfies the Pontryagin Maximum Principle (MP), the collection of Lagrange multipliers, for simplicity, being unique, up to normalization. We assume that w^0 is totally singular, which means that Pontryagin function does not depend on $u \in U$, i.e. $\psi(t)F(x^0(t),t) = 0$. Furthermore, we assume that $u^0(t)$ is continuous, and for all t lies in the relative boundary of one and the same face U_0 of U. (The case $U_0 = U$ is not forbidden.) The case $U_0 \neq U$ is the simplest case of a boundary control.

There is a large number of works on higher-order optimality conditions for Problem A, done, in some special statements, since early 1960-s (Kelley, Kopp, Moyer, Bryson, Robbins, Goh, Speyer, Jacobson, Bell, McDanell, Powers, Gabasov, Kirillova, Krener, Agrachev, Gamkrelidze, Milyutin, Knobloch, Zelikin, Gurman, Dykhta, Lamnabhi-Lagarrigue, Stefani and others). An overwhelming majority of conditions obtained are higher-order *necessary* conditions which have *a pointwise* character (like the classical Legendre condition). Only few works are devoted to *sufficient* conditions, but these conditions obtained till recently are rather far from necessary ones.
In [1, 2] and his earlier works, the author gave for Problem A both necessary and close to them sufficient second order conditions of optimality (*adjoint pairs* of conditions). Here we specify these conditions in a form, convenient for the above case of a boundary control. A difficulty in this case is that one cannot take two-side variations of the control, which are of crucial importance in obtaining higher-order optimality conditions for an interior control.

Speaking of "optimality", we consider two types of minimum - a classical weak and a Pontryagin minimum (Π-minimum), which is an L_1-minimum with respect to control on any uniformly bounded control set. Note that Π-minimum allows one to take so-called needle-type variations of the control.
Speaking of "second order" conditions, we chose the following quadratic functional (order) of estimation:

$$\gamma(\bar{w}) = |\,\bar{x}(t_0)\,|^2 + |\,\bar{y}(t_1)\,|^2 + \int |\,\bar{y}(t)\,|^2\, dt, \qquad \text{where} \qquad \dot{\bar{y}} = \bar{u}, \qquad \bar{y}(t_0) = 0.$$

Let $\Omega(\bar{w})$ be the second variation of Lagrange function at w^0. Denote $f_0(x,t) = f(x,t) + \sum u_i^0 f_i(x)$ and let $\mathcal{K}$ be the cone of critical variations for Problem A with unbounded control, i.e. the set of all $\bar{w} = (\bar{x}, \bar{u})$ such that

$$\varphi_i'(p^0)\,\bar{p} \le 0 \quad \text{for all active} \quad i \in \{0,\dots,\nu\}, \qquad g'(p^0)\,\bar{p} = 0, \qquad \dot{\bar{x}} \;=\; f_0'(x^0)\,\bar{x} \;+\; \sum \bar{u}_i\, f_i(x^0)$$

(the prime at f_0 stands for the derivative w.r.t. x along $x^0(t)$).
Denote $N = \operatorname{con}(U - u^0(t))$. The critical cone for Problem A is $\mathcal{K} \cap \mathcal{N}$, where $\mathcal{N} = \{\bar{w} \mid \bar{u}(t) \in N \ a.e.\}$.

Theorem 1. *a) Let w^0 be a point of a weak minimum (Π-minimum) in Problem A. Then some pointwise conditions G_0 (G_0 plus E_0, resp.) hold, and moreover, $\Omega(\bar{w}) \ge 0$ for all $\bar{w} \in \mathcal{K} \cap \mathcal{N}$.*
b) Suppose that some pointwise conditions G_a (G_a plus E_a) hold for an $a > 0$, and

$$\Omega(\bar{w}) \;\ge\; a\,\gamma(\bar{w}) \qquad \text{for all} \quad \bar{w} \in \mathcal{K} \cap \mathcal{N}. \tag{3}$$

Then w^0 is a point of a strict weak minimum (strict Π-minimum, resp.) in Problem A.

These necessary and sufficient conditions are close to each other; we call them adjoint pairs of conditions. In this sense they are similar to those in the analysis and the classical calculus of variations.

Conditions $G_a, a \geq 0$ concern the quadratic form Ω. Condition E_a includes also coefficients of the third variation of Lagrange function and the admissible control set U.

Using the known Goh transformation $\quad \bar{x} = \bar{\xi} + \sum \bar{y}_i\, f_i(x^0(t)), \quad$ one can present Ω $\quad$ in the form:

$$\Omega(\bar{\xi},\bar{y},\bar{u}) = \int_{t_0}^{T} \left(-\frac{1}{2}\,\psi(f_0''\bar{\xi},\bar{\xi}) \;+\; \sum_i \bar{y}_i\,\psi\,[f_i,f_0]'\bar{\xi} \;+\; \frac{1}{2}\sum_{i,j}\bar{y}_i\bar{y}_j\,\psi\,[\,[f_i,f_0],f_j] \;+\; \frac{1}{2}\sum_{i,j}\bar{y}_i\bar{u}_j\,\psi\,[f_i,f_j] \right) dt \;+$$

$$+\,\omega(\bar{\xi}_0,\bar{\xi}_T,\bar{y}_T) \;+\; \left\{ \frac{1}{2}\sum_{i,j}\bar{y}_i\bar{y}_j\,(\psi, f_i' f_j) \;+\; \sum_i \bar{y}_i\,(\psi, f_i'\bar{\xi}) \right\}\Bigg|_T , \qquad \dot{\bar{\xi}} = f_0'\,\bar{\xi} \;+\; \sum_i \bar{y}_i\,[f_0,f_j],$$

where ω is the second variation of the terminal Lagrange function, and $[\,,\,]$ are Lie brackets.

Now let us write out conditions G_a and E_a. Denote by $l(N)$ the maximal linear subspace in N, and by $l'(N)$ a complementary subspace. Then $N_1 = N \cap l'(N)$ is a sharp cone, and $N = N_1 \oplus l(N)$, so that for all $u \in N$ we have $u = (u',u'')$, $u' \in N_1$, $u'' \in l(N)$. Taking $l(N)$ to be the coordinate subspace for indices $\{r+1,\ldots,k\}$, and $l'(N)$ that for $\{1,\ldots,r\}$, we have $u' = (u_1,\ldots,u_r) \in N_1$, and $u'' = (u_{r+1},\ldots,u_k)$ is a free variable.
We say that $\psi(t)$ satisfies conditions G_a, for some $a \geq 0$, if the following three conditions hold along $x^0(t)$:

$$i) \quad \psi(t)\,[f_i,f_j] = 0 \qquad \forall\, i,j \geq r+1, \tag{4}$$

$$ii) \text{ the symmetric } (k-r)\times(k-r) \text{ matrix } \quad Q(t) = \|\, q_{ij}(t) = \psi\,[\,[f_i,f_0],f_j]\,\| \;\geq\; a \tag{5}$$

$$iii) \text{ condition (4) holds for all } \quad i \leq r, \quad j \geq r+1. \tag{6}$$

Conditions (4) and (5) concerning the free controls are known Goh conditions; the boundary condition (6) is a new one. To write out condition E_a, we need the following qubic functional:

$$e(\bar{y}) = e(\bar{y}'') = \int (-\frac{1}{2}\sum \bar{u}_i\bar{y}_j\bar{y}_s\,\psi(f_i''f_j,f_s) \;+\sum \bar{y}_i\bar{u}_j\bar{y}_s\,\psi(f_i'f_j'f_s)\,)\,dt \;=\int (\,(\mathcal{E}(t)\,\bar{y},\bar{y},\bar{u})\,)\,dt,$$

both sums being taken over $i,j,s > r$, $\quad$ thus $\mathcal{E}(t)$ being a $(k-r)\times(k-r)\times(k-r)$ tensor.
We say that $\psi(t)$ satisfies condition E_a, if for any $t_* \in [t_0,T]$, for any Lipschitz function $\bar{y}(t)$, having $\bar{y}(t_0) = \bar{y}(T) = 0$ and $\dot{\bar{y}} = \bar{u} \in U_0 - u^0(t_*)$, the following inequality holds:

$$L\,[\psi,t_*](y) \;=\; \int (\,(Q(t_*)\bar{y},\bar{y}) + (\mathcal{E}(t_*)\bar{y},\bar{y},\bar{u})\,)\,dt \;\geq\; a\int (\bar{y},\bar{y})\,dt. \tag{7}$$

Theorem 2. *If (5) holds for $a > 0$, $0 < a' < a$, and U_0 is small enough, then condition $E_{a'}$ holds, and, however large U is, inequality (3) guarantees a Π-minimum at w^0 in Problem A.*

For the simplest case, when $U_0 = \mathbb{R}^{k-r}$, condition E_a decomposes into (5) and the following equations:
$\forall\, i,j,s > r \qquad \psi(t)\,[\,[f_i,f_j],f_s] = 0 \quad$ along $x^0(t)$.

Theorems 1 and 2 remain as well valid, in appropriate forms, for the general case, when Lagrange multipliers are not unique. They proved to be helpful in the problem of minimality of abnormal sub-Riemannian geodesics.

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1.References

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