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Existence Theorem for Optimal Control Problems on an Infinite Time Interval

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We consider an optimal control problem on an infinite time interval. The system is linear in the control, the cost functional is convex in the control, and the control set is convex and compact. We propose a new condition on the behavior of the cost at infinity, which is weaker than the previously known conditions, and prove the existence theorem under this condition. We consider several special cases and propose a general abstract scheme.

1 Introduction

Optimal control problems on infinite time interval appear both in theoretical and in applied fields of mathematics, e.g., in dynamical models of mathematical economics [1]–[7]. The existence of solutions in such problems cannot be proved so easily as in the problems on a finite time interval [8]–[10]. The existence conditions suggested in the known papers on this subject either are too restrictive or have difficult formulation, and hence cannot be verified easily. In this paper we consider a broad class of problems including, in particular, most of economic dynamical problems, and propose rather natural conditions guaranteeing the existence of solutions in these problems, which are weaker than the previously known conditions.

2 Statement of the problem and assumptions

On the half-axis $[0,\infty)$ we consider n-vector functions $x(t)$ absolutely continuous on each interval $[0,T]$ (we write $x(\cdot)\in AC[0,\infty)$) and measurable r-vector functions $u(t)$ which are essentially bounded on each $[0,T]$ (the space of these functions we denote by $L_\infty[0,\infty)$).

Note that the functions from $AC[0,\infty)$ can be not absolute continuous on the whole half-axis $[0,\infty)$. The similar concerns also the functions from $L_\infty[0,\infty)$. (We do not use more specific notation for the spaces AC and L_∞ on $[0,\infty)$, since these spaces will be understood here only in the above sense.)

On the space of pairs of functions (x,u) we consider the following optimal control problem:

$$J(x,u) = \beta(x(0)) + \int_0^\infty \varphi(t,x,u)\,dt \;\to \min, \tag{1}$$

$$\dot x = f(t,x,u), \tag{2}$$

$$x(0) \in M_0\,, \tag{3}$$

$$u(t) \in U(t,x(t))\,, \tag{4}$$

$$x(t) \in S(t)\,. \tag{5}$$

A pair of functions $(x,u) \in AC \times L_\infty[0,\infty)$ satisfying constraints (2)–(5) on $[0,\infty)$ is called admissible (as usual, all relations with measurable functions are assumed to be satisfied almost everywhere), and the set of all admissible pairs will be denoted by Ω. The cost functional (1) is considered for all pairs $(x,u) \in \Omega$ for which the corresponding Lebesgue integral exists on any interval $[0,T]$ and converges to a finite or infinite limit as $T \to \infty$. Thus,

$$J(x,u) = \lim_{T\to\infty} J_T(x,u), \qquad \text{where} \quad J_T(x,u) = \beta(x(0)) + \int_0^T \varphi(t,x,u)\,dt,$$

and the limit is meant in the above extended sense. (Below, we give a condition guaranteeing the convergence of the integral for any pair $(x,u) \in \Omega$.)

Remark. In order to avoid the "inconvenient" question about the convergence of integral in (1), some authors suggest to change the concept of optimality itself, i.e., the very principle of comparison of two admissible pairs: to compare not the limit values of integral (1) (which may not exist), but instead, to consider the behavior of the difference $J_T(x'',u'') - J_T(x',u')$ of the functionals on the intervals $[0,T]$ as $T \to \infty$. However, this approach does not provide a natural unique definition of which pair generates a "better" family of values J_T, and so, one must consider several different definitions of this and, respectively, several different concepts of optimality (see the papers [4]–[7] and references therein). In this paper we do not consider these generalizations. The admissible pairs are compared here in the usual way, by the values of the cost functional, and therefore, the optimality is understood as the attainment of the minimal possible value of the functional.

Assumptions:

A1) The function $f(t,x,u)$ is continuous with respect to the pair (t,x) and linear in u , i.e., $f(t,x,u) = a(t,x) + B(t,x)u,$ where the $n-$ dimensional vector a and $n \times r-$ matrix B continuously depend on (t,x) ;

A2) the set $S(t)$ is closed $\forall\, t \geq 0,$ whereas its dependence on t is arbitrary;

A3) the set M_0 is compact in $\mathbb{R}^n$;

A4) the set-valued mapping $U : \mathbb{R}^{1+n} \to \mathbb{R}^r$ is upper semicontinuous, and $U(t,x)$ is a convex compact set for all (t,x) ;

A5) the function f together with the mappings S and U satisfies the Filippov condition [8], i.e., there exists a number c such that $\forall\, t \geq 0,\ \forall\, x \in S(t),\ \ \forall\, u \in U(t,x)$

$$(x, f(t,x,u)) \leq c(|x|^2 + 1);$$

A6) the function $\beta(\cdot)$ is continuous on M_0 ;

A7) the function $\varphi(t,x,u)$ is continuous in (t,x,u) and convex in u .

(Note that, if $S(t)$ and $U(t,x)$ are uniformly bounded, then Assumption A5 is a priori satisfied. However, the fulfilment of A5, both in this case and in the general case, does not yet guarantee the existence of an admissible trajectory of system (2)–(5) on a given interval $[0,T],$ the more so on $[0,\infty).$ The existence of an admissible trajectory will be, as always is in the existence theorems for extremal problems, not proved, but just postulated.)

All these assumptions are quite standard in the existence theorems for problems on a fixed finite time interval. Besides of them, we make one more assumption about the behavior of the family of functions $\varphi(t,x(t),u(t))$ at infinity.

For any number a , set $a^+ = \max{(a,0)},$ and $a^- = \max{(-a,0)}$ (both these values are nonnegative), so that $a = a^+ - a^-$.

By *a portion* of the functional J on an interval $[T',T''],$ let us call the number

$$\int_{T'}^{T''} \varphi(t,x(t),u(t))\,dt\,.$$

The following assumption plays a key role in our considerations.

Assumption A8. The negative parts of the portions of the cost functional tend to zero as $T',\, T'' \to \infty,\ \ T' < T'',$ uniformly over all admissible trajectories.

In other words, for any $\varepsilon > 0$ there exists a T_ε such that $\forall\, T'' > T' > T_\varepsilon$, $\forall\,(x,u) \in \Omega$

$$\int\limits_{T'}^{T''} \varphi(t, x(t), u(t))\, dt \;>\; -\varepsilon\,.$$

This condition is obviously equivalent to the following one: there exists a function $\alpha(T) \to 0+$ as $T \to \infty$, and a number T_0 such that $\forall\, T > T_0$, $\forall\, T'' > T' \geq T$, and $\forall\,(x,u) \in \Omega$

$$\int\limits_{T'}^{T''} \varphi(t, x(t), u(t))\, dt \;\geq\; -\alpha(T)\,. \tag{6}$$

Below, we will point out some cases when this assumption is a priori fulfilled.

Let us show that the fulfilment of Assumption A8 on Ω guarantees the convergence of the integral (in the above extended sense) for any pair $(x,u) \in \Omega$. We will use the following simple fact.

Lemma 1. *Let a numerical sequence γ_k have no limit (neither finite, nor infinite). Then, there exist numbers $z_1 < z_2$ such that $\forall\, K$ there are $k_1,\ k_2 > K$, for which $\gamma_{k_1} < z_1$ and $\gamma_{k_2} > z_2$.*

Proof. Since the sequence γ_k has no limit, it contains two subsequences converging to different limits: $\gamma_{k_s^1} \to c_1$ and $\gamma_{k_s^2} \to c_2$, where $c_1 < c_2$ (with possible cases $c_1 = -\infty$, $c_2 = +\infty$). Take arbitrary $z_1,\ z_2 \in \mathbb{R}$ such that $c_1 < z_1 < z_2 < c_2$. Then, $\forall\, K$ there exists a number $k_1 > K$ in the subsequence k_s^1 and a number $k_2 > K$ in the subsequence k_s^2 such that $\gamma_{k_1} < z_1$, $\gamma_{k_2} > z_2$. The lemma is proved. $\square$

Lemma 2. *Suppose that the functional J satisfies Assumption A8. Then, for all $(x,u) \in \Omega$ the corresponding integral converges either to a finite limit or to $+\infty$.*

Proof. Suppose first that, for a pair $(x,u) \in \Omega$, the integral does not converge (in our extended sense), i.e., the limit

$$\lim_{T\to\infty} \int\limits_0^T \varphi(t, x(t), u(t))\, dt \tag{7}$$

does not exist. Hence this limit also does not exist for some countable sequence $T \to \infty$. Then, by Lemma 1, there exist numbers $z_1 > z_2$ such that, for any T

there are $T_1 > T$ and $T_2 > T_1$ in this countable sequence such that $J_{T_1}(x,u) > z_1$ and $J_{T_2}(x,u) < z_2$, and hence

$$\int\limits_{T_1}^{T_2} \varphi(t, x(t), u(t))\, dt \;=\; J_{T_2}(x,u) - J_{T_1}(x,u) \;<\; z_2 - z_1 \;= const < 0\,.$$

But this contradicts Assumption A8. Hence our supposition is not true.

Now consider the case in which the limit (7) is equal to $-\infty$. Then for any T', there is a $T'' > T'$ such that

$$\int_0^{T''} \varphi(t, x(t), u(t))\, dt \;<\; \int_0^{T'} \varphi(t, x(t), u(t))\, dt \;-1,$$

and hence

$$\int_{T'}^{T''} \varphi(t, x(t), u(t))\, dt < -1,$$

i.e., for $\varepsilon = 1$, we again obtain a contradiction to A8. The lemma is proved. □

Thus, for any admissible pairs (x,u), the quantity

$$J(x,u) \;=\; \lim_T J_T(x,u)$$

is well defined and J always is either finite or equal to $+\infty$. Moreover, it follows from the proof of Lemma 2 that A8 is the most natural assumption ensuring these properties of the functional J.

Theorem 1 (the main theorem). *Suppose that, under the above Assumptions A1–A8, there is at least one pair $(x,u) \in \Omega$ for which $J(x,u) < +\infty$. Then there exists a pair $(x_0, u_0) \in \Omega$ at which the functional attains its minimal value (i.e., the problem under study has a solution).*

To prove this assertion, we need several properties of the functional J and of the set of admissible trajectories on a fixed interval $[0,T]$.

3 Several properties for a fixed interval

Lemma 3. *Let $U(y)$ be an upper semicontinuous set-valued mapping $I\!R^m \to I\!R^r$ with compact values. Then, on any compact (and hence on any bounded) set of y, the values $U(y)$ are uniformly bounded, i.e., for any compact set $K \subset I\!R^m$, there exist a constant R such that the set $U(y)$ is contained in the ball $B_R(0)$ for any $y \in K$.*

Proof. Since $U(y)$ is an upper semicontinuous mapping, for any y there exists a neighborhood $\mathcal{O}(y)$ of y such that the inclusion $U(y') \subset U(y) + B_1(0)$ holds for any $y' \in \mathcal{O}(y)$. The union of these neighborhoods $\mathcal{O}(y)$ over all $y \in K$ covers the entire compactum K, and, by the definition of a compact set, a finite subcovering can be chosen from this covering. Namely, there exist finitely many points $y_1, \ldots, y_m \in K$ and their neighborhoods $\mathcal{O}(y_i)$ such that the set $U(y')$ is contained in $U(y_i) + B_1(0)$ for any $y' \in \mathcal{O}(y_i)$, and these neighborhoods cover the entire compactum K.

The union V of the bounded sets $U(y_i) + B_1(0)$ over all $i = 1, \ldots, m$ is also bounded, i.e., it is entirely contained in the ball $B_R(0)$ for some R. Since for any $y \in K$, there exists a number i such that $y \in \mathcal{O}(y_i)$, we have $U(y) \subset U(y_i) + B_1(0) \subset V$, and hence $U(y) \subset B_R(0)$. $\square$

Corollary. *Suppose that $U(t,x)$ is an upper semicontinuous mapping $\mathbb{R}^{1+n} \to \mathbb{R}^r$ with compact values. Then, for any $T > 0$ and any bounded set $Q \subset \mathbb{R}^n$, there is an $R = R(T,Q)$ such that the inclusion $U(t,x) \subset B_R(0)$ holds for any $t \in [0,T]$ and any $x \in Q$.*

Proof. One should apply Lemma 3 to the mapping $U(y)$, where $y = (t,x)$, and to the compact set $K = [0,T] \times \overline{Q}$. $\square$

Lemma 4. *Let $U(t,x)$ be an upper semicontinuous mapping with compact values, let the function $f(t,x,u)$ together with mappings S and U satisfy A5 (i.e., the Filippov condition), and let $M_0 \subset \mathbb{R}^n$ be a bounded set. Then, for any T, there are constants D_T, D'_T, R_T such that the estimates*

$$|x(t)| \le D_T\,, \qquad |\dot{x}(t)| \le D'_T\,, \qquad |u(t)| \le R_T \tag{8}$$

hold almost everywhere on $[0,T]$ for any solution to system (2)–(5).

Proof. Consider the function $z(t) = |x(t)|^2 + 1$. For this function, we have $\dot{z} = 2(x,f)$ on the trajectories of system (2), and hence it follows from A5 that $\dot{z} \le 2c(|x|^2+1)$, i.e., $\dot{z} \le 2cz$. Since $z(t) \ge 0$, we obtain the inequality $z(t) \le z(0)e^{2ct}$ for any $t \ge 0$. If $|M_0| \le r$, then we have $z(t) \le (r^2+1)\,e^{2cT}$ on the interval $[0,T]$, which implies the first desired estimate in (8).

Since $|x(t)| \le D_T$, by corollary of Lemma 3 we have $|u(t)| \le R_T$ with a constant R_T; therefore, the triple $(t, x(t), u(t))$ always belongs to the compact set $[0,T] \times B_{D_T}(0) \times B_{R_T}(0)$. Since the function f is continuous, it is also bounded on this compact set by a constant D'_T. Thus, $|\dot{x}(t)| \le D'_T$. The lemma is proved. $\square$

Thus, the set of solutions $x(t)$ to system (2)–(5) is uniformly bounded and uniformly Lipschitz continuous, and the set of controls $u(t)$ is uniformly bounded. Hence, by the Ascoli-Arzela and Alaoglu theorems [13], we obtain the following

Corollary. *The set of solutions $x(t)$ to system (2)–(5) is precompact in the space $C[0,T]$, and the corresponding set of controls $u(t)$ is precompact in the space $L_\infty[0,T]$ in the weak-* topology (i.e., in the topology determined by functionals from $L_1[0,T]$).*

Let Ω_T be a set of all $(x,u) \in AC[0,T] \times L_\infty[0,T]$ satisfying relations (2)–(5) on $[0,T]$. It follows from the above corollary that Ω_T is a precompact set with respect to the product of the uniform and weak-* topologies. As is known, the weak-* topology on bounded sets in the space $L_\infty[0,T]$ is metrizable (because L_1 is a separable space), and hence, to study this topology, it suffices to consider converging sequences.

Lemma 5. *The set Ω_T is closed w.r.t. the uniform convergence of $x(t)$ and the weak-* convergence of $u(t)$; namely, if $(x_k,u_k) \in \Omega_T$, $x_k \Longrightarrow x_0$, $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$, then $(x_0,u_0) \in \Omega_T$.*

Proof. The closedness of constraints $x(0) \in M_0$ and $x(t) \in S(t)$ obviously follows from the fact that the sets M_0 and $S(t)$ for all $t \geq 0$ are closed. The closedness of constraint $\dot{x} = f(t,x,u)$ can be easily obtained from the linearity of f in u if one passes to the integral form of this equation.

The main difficulty is to show that the inclusion $u(t) \in U(t,x(t))$ is also closed. This is a nontrivial fact, which is of intrinsic interest. In order not to go far away from the main topic, we do not prove this fact here. We only note that it follows from the closedness of graph of the mapping $U(t,x)$ (which follows from its upper semicontinuity) and the convexity of its values (e.g., see [10, Secs. 8.5 and 10.6] for details). The lemma is proved. $\square$

Summarizing the facts obtained, we arrive at the following assertion.

Lemma 6. *In the space of pairs $(x,u) \in AC[0,T] \times L_\infty[0,T]$, the set Ω_T is a metrizable compactum in the product of the topology generated by the norm $||x||_C$ and the weak-* topology of u .*

Now, we turn to the functional J_T assumed to be defined on the space $C[0,T] \times L_\infty[0,T]$.

Lemma 7. *Let a function φ satisfy Assumption A7, and let be given sequences $x_k \Longrightarrow x_0$ (uniformly) and $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$ (weakly in L_∞ with respect to L_1) on the interval $[0,T]$. Then*

$$\int_0^T \varphi(t,x_0(t),u_0(t))dt \;\leq\; \liminf_{k\to\infty} \int_0^T \varphi(t,x_k(t),u_k(t))dt,$$

i.e., functional $\int_0^T \varphi(t,x,u)\,dt$ *is lower semicontinuous with respect to this convergence.*

This fact is widely known (e.g., see [9]–[12].) This and the continuity of $\beta(x)$ imply that our functional J_T is also lower semicontinuous with respect to the above convergence.

4 Proof of the main theorem

Now, we return to the problem of infinite interval. Introduce a convergence in the function spaces on $[0,\infty)$.

Let $C[0,\infty)$ be the space of all continuous n -dimensional functions on $[0,\infty)$. We shall write $x_k \Longrightarrow x_0$ for the elements of this space if x_k uniformly converges to x_0 on any interval $[0,T]$.

We shall write $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$ for the elements of the space $L_\infty[0,\infty)$ introduced above if u_k weakly-* converges to u_0 on any interval $[0,T]$.

The admissible set Ω can be treated as a subset of the space $C[0,\infty)\times L_\infty[0,\infty)$. As follows from its definition, Ω consists of all the pairs (x,u) whose restriction to any interval $[0,T]$ belongs to the corresponding set Ω_T .

Lemma 8. *The admissible set* Ω *is closed with respect to the above convergence, i.e., if* $(x_k,u_k)\in\Omega$, $x_k \Longrightarrow x_0$, *and* $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$, *then* $(x_0,u_0)\in\Omega$.

Proof. By definition, for any $T>0$ we have $(x_k,u_k)\in\Omega_T$, $x_k \Longrightarrow x_0$, and $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$ on the interval $[0,T]$. Since the set Ω_T is closed by Lemma 5, we have $(x_0,u_0)\in\Omega_T$. This means that the inclusion $(x_0,u_0)\in\Omega$ also holds on the entire half-line. The lemma is proved. $\square$

Let us introduce one more definition, which is convenient to use in what follows. The quantity

$$\Theta_T(x,u) = \int\limits_T^\infty \varphi(t,x(t),u(t))\,dt$$

will be called the *tail* of the functional J on the interval (T,∞). This quantity is well defined, because, under Assumption A8, the integral converges (in the extended sense) for any pair $(x,u)\in\Omega$.

Lemma 9. *If Assumption A8 is satisfied on* Ω, *then the negative parts of the tails of the functional tend to 0 as* $T\to\infty$ *uniformly over all* $(x,u)\in\Omega$.

Moreover, for the same function $\alpha(T)$ *and the same* T_0 *(appearing in A8), the estimate*

$$\Theta_T(x,u) \ \geq \ -\alpha(T) \tag{9}$$

holds for all $(x,u) \in \Omega$ *and all* $T \geq T_0$.

Proof. Suppose that A8 is satisfied, i.e., there is T_0 and $\alpha(T) \to 0+$ such that, for all $T > T_0$ and all $T'' > T' \geq T$, the estimate

$$\int\limits_{T'}^{T''} \varphi(t, x(t), u(t))\, dt \ \geq \ -\alpha(T)$$

holds for any pair $(x,u) \in \Omega$. Then, fixing T' and letting T'' tend to $+\infty$, we obtain the desired inequality (9). □

Now, we establish an analog of Lemma 7 concerning the lower semicontinuity of the functional for the infinite interval.

Lemma 10. *Let Assumptions A1–A8 be satisfied. Then the functional* J *is lower semicontinuous on* Ω *in the above convergence, i.e., if* $(x_k, u_k) \in \Omega$, $x_k \Longrightarrow x_0$, *and* $u_k \overset{\text{weak}-*}{\longrightarrow} u_0$, *then*

$$J(x_0, u_0) \ \leq \ \liminf_{k\to\infty} J(x_k, u_k)\,.$$

Proof. First, by Lemma 8, $(x_0, u_0) \in \Omega$. Next, let $\liminf\limits_{k\to\infty} J(x_k, u_k) = \mu_0$. We must show that $J(x_0, u_0) \leq \mu_0$.

Assume again that $\beta(\cdot) \equiv 0$. For brevity, denote $\varphi_k(t) = \varphi(t, x_k(t), u_k(t))$. For any T, the functional $J(x_k, u_k)$ can be represented in the form

$$J(x_k, u_k) = \int\limits_0^T \varphi_k(t)\, dt \ + \Theta_T(x_k, u_k).$$

Since J satisfied Assumption A8, Lemma 9 says that, for any $T \geq T_0$, we have $\Theta_T \geq -\alpha(T)$, and so

$$\int\limits_0^T \varphi_k(t)\, dt \ = J(x_k, u_k) - \Theta_T(x_k, u_k) \leq \ J(x_k, u_k) + \alpha(T)\,.$$

Hence, for any fixed T, we have

$$\liminf_{k\to\infty} \int_0^T \varphi_k(t)\, dt \ \leq \ \liminf_{k\to\infty} J(x_k, u_k) + \alpha(T) \ = \ \mu_0 + \alpha(T)\,.$$

By Lemma 7 , this implies

$$\int_0^T \varphi_0(t)\,dt \ \leq \mu_0 + \alpha(T)\,.$$

Now, letting $T \to \infty$ and taking into account Lemma 2, we finally obtain

$$J(x_0, u_0) = \lim_{T\to\infty} \int_0^T \varphi_0(t)\,dt \ \leq \mu_0\ .$$

The lemma is proved. □

The next example shows that Assumption A8 in this lemma is essential: if one neglects it, the lemma ceases to be true.

Example. Consider the sequence of functions $u_k(t)$ equal to -1 for $t \in [k, k+1]$ and to 0 in other cases. This sequence weakly-* converges in our sense to $u_0 \equiv 0$. Set

$$\varphi(t,x,u) = u, \quad \beta(x) = 0, \quad f(t,x,u) = 0, \quad x(0) = 0, \quad U(t,x) = [-1, 0].$$

Then all the assumptions of Lemma 10, except for Assumption A8, are satisfied. But, in this case, the inequality

$$\int_0^\infty \varphi(u_0(t))\,dt \ \leq \ \liminf_{k\to\infty} \int_0^\infty \varphi(u_k(t))\,dt$$

does not hold, because the left-hand side equals zero, while, for all k, the integral on the right-hand side is

$$\int_0^\infty \varphi(u_k(t))\,dt \ = \ \int_k^{k+1} (-1)\,dt \ = -1.$$

Proof of the main theorem. Take an arbitrary minimizing sequence

$$(x_k, u_k) \in \Omega : \qquad J(x_k, u_k) \ \to inf J = J_*.$$

We must show that there exists a pair $(x, u) \in \Omega$ such that $J(x, u) = J_*$.

(1) Take an arbitrary $T_1 > 0$. By Lemma 6 , the set Ω_{T_1} is metrizable and compact. Hence, the sequence (x_k, u_k) contains a subsequence (x_k^1, u_k^1) converging on $[0, T_1]$ to a pair $(x_0^1, u_0^1) \in \Omega_{T_1}$. (Saying that x_k^1 is a subsequence of the sequence x_k, we mean that there exists an increasing sequence of integers $n_k \to \infty$ such that $x_k^1 = x_{n_k}$ for any k.)

(2) Take an arbitrary $T_2 > T_1+1$. As in the preceding case, the sequence (x_k^1, u_k^1) contains a subsequence (x_k^2, u_k^2) converging on $[0, T_2]$ to a pair $(x_0^2, u_0^2) \in \Omega_{T_2}$.

Then, since (x_k^2, u_k^2) is a subsequence of the sequence (x_k^1, u_k^1), and their limits coincide on $[0, T_1]$, namely, $x_0^2 \equiv x_0^1$ and $u_0^2 \equiv u_0^1$, i.e., the new limit pair is an extension of the old pair to the interval $[T_1, T_2]$.

(3) Next, take an arbitrary $T_3 > T_2+1$ and choose a subsequence (x_k^3, u_k^3) from the sequence (x_k^2, u_k^2), and so on.

Thus, at the m th step of this procedure, we have $T_m > T_{m-1}+1$ and a sequence (x_k^m, u_k^m) converging to a pair $(x_0^m, u_0^m) \in \Omega_{T_m}$, which coincides with (x_0^{m-1}, u_0^{m-1}) on the preceding interval $[0, T_{m-1}]$.

Now, define the pair $(x_0, u_0) \in AC \times L_\infty[0, \infty)$ coinciding with (x_0^m, u_0^m) on each interval $[0, T_m]$. This pair is well defined because of the coincidence $(x_0^k, u_0^k) \equiv (x_0^m, u_0^m)$ on the interval $[0, T_m]$ for any $k > m$. Then, on every $[0, T_m]$ we have $(x_0, u_0) \in \Omega_{T_m}$ which, by the definition of Ω, implies that $(x_0, u_0) \in \Omega$ on the entire half-line.

(4) Consider the diagonal sequence (x_k^k, u_k^k). Obviously, it has the following property: for any fixed m and all $k \geq m$, this sequence is contained in the subsequence (x_i^m, u_i^m), $i = 1, 2, ...,$ and hence it converges to $(x_0^m, u_0^m) = (x_0, u_0)$ on the interval $[0, T_m]$. And since $T_m \to \infty$, the sequence (x_k^k, u_k^k) converges to (x_0, u_0) on any interval $[0, T]$, i.e., by definition, it converges in the space $C[0, \infty) \times L_\infty[0, \infty)$.

(5) Finally, consider the functional $J(x_k^k, u_k^k)$. By Lemma 10, it is lower semicontinuous w.r.t. the above convergence; therefore,

$$J(x_0, u_0) \leq \liminf_{k \to \infty} J(x_k^k, u_k^k) = J_* ,$$

and the inequality $J(x_0, u_0) < J_*$ is impossible, since $J_* = inf\, J$. Hence we obtain $J(x_0, u_0) = J_*$, Q.E.D. $\square$

Remark. Since, for any $T > 0$, the convergence in Ω_T can be determined by a certain metric ρ_T , the convergence in Ω can also be given by a metric, for example, chosen in the form of standard combination

$$\rho((x', u'), (x'', u'')) \;=\; \sum_{m=1}^{\infty} \frac{1}{2^m}\, F\left(\rho_{T_m}((x', u'), (x'', u''))\right),$$

where

$$T_m \to \infty \qquad \text{and} \qquad F(z) = \frac{z}{1+z}\,.$$

In steps (1)–(4) of the above proof, we actually proved that each sequence of elements of Ω contains a converging subsequence, i.e., that Ω is a compact set in this metric.

Thus, our main theorem completely fits in the framework of the general Weierstrass principle: a lower semicontinuous function on a compact set attains its infimum.

5 Special cases

We present several conditions under which Assumption A8 is satisfied.

1. The simplest condition, which is mentioned in many papers and which, indeed, is most often satisfied in special problems, is that $\varphi(t,x,u)$ is bounded below by an integrable function, i.e., there exists an $l(t) \in L_1(0,\infty)$, $l(t) \ge 0$, such that

$$\varphi(t,x(t),u(t)) \ge -l(t) \qquad \text{a.e. on} \quad (0,\infty)$$

for any pair $(x,u) \in \Omega$. In this case, we have an obvious estimate for the functional portions,

$$\int_{T'}^{T''} \varphi(t,x(t),u(t))\,dt \ \ge \int_{T'}^{T''} -l(t)\,dt \ge \int_{T'}^{\infty} -l(t)\,dt \ = -\alpha(T'),$$

and since $\alpha(T') \to 0$ as $T' \to \infty$, our condition (6) is satisfied.

2. Suppose that all the constraints in the problem are independent of t, and the functional has the form

$$J = \ \beta(x(0)) + \int_0^{\infty} e^{-rt} L(x,u)\,dt \ \to \max, \tag{10}$$

where the function L is continuous in (x,u) and concave (i.e., upper convex) in u, and r is a positive number (it is called *the discount factor*). Such problems are typical of dynamical models in mathematical economics [1]–[7].

Let us assume that there exist nonnegative numbers p, h, q, C, and K such that the estimates

$$(x, f(x,u)) \ \le \ p\,(|x|^2 + h), \tag{11}$$

$$L(x,u) \ \le \ C|x|^q + K \tag{12}$$

hold for all $x \in S$ and $u \in U(x)$. Let also $p\,q < r$. Then the function

$$\varphi(t,x,u) = e^{-rt} L(x,u)$$

is bounded from above by an integrable function and hence, according to the preceding case 1, Assumption 8 is satisfied for the maximization problem.

Indeed, consider the function $z(t) = |x(t)|^2 + h$. It follows from (11) that $|\dot{z}| \le 2\,pz$, hence $z(t) \le z(0)\, e^{2p\,t}$, and therefore, $|x(t)| \le C_1\, e^{pt}$ with some constant C_1. Then we have $L(x,u) \le C_2\, e^{pq\,t} + K$ with a constant C_2, and so,

$$e^{-rt} L(x,u) \;\le\; C_2\, e^{(pq-r)t} + K\, e^{-rt} \;= l(t),$$

where the function $l(t)$ is integrable on $(0,\infty)$, because $pq - r < 0$.

3. Consider also the case where the problem has the same form as in case 2, but the controlled system is linear (with constant coefficients), $\dot{x} = Ax + Bu$, and the control set U is independent of x. Suppose that there is a number p such that all the eigenvalues λ of the matrix A have $Re\,\lambda < p$ and that there are nonnegative numbers q, C, and K such that, for all $x \in S$ and $u \in U$ the estimate (12) holds, and moreover, $pq < r$. Then the family of functions $\varphi(t,x,u) = e^{-rt} L(x,u)$ is again bounded from above by an integrable function, and hence Assumption A8 is satisfied.

Indeed, as is known in this case, there is a constant C_1 such that any solution to the system $\dot{x} = Ax + Bu$, $u \in U$, with any initial value $x(0) \in M_0$ satisfies the estimate $|x(t)| \le C_1\, e^{pt}$. Next, one must repeat the corresponding argument of the preceding case.

6 Comparison with well-known results

The most general results concerning this problem were obtained by Balder [3]. Let us compare his and our assumptions about the behavior of the functional at infinity. In Balder's paper, the notion of strong uniform integrability of a family of functions was introduced, and it was assumed that the family $\{\varphi^-(t, x(t), u(t))\}$ has this property.

Let M be a measurable set on the real line. By $L_1(M)$ we denote the space of measurable Lebesgue integrable functions, and by $L_1^+(M)$ we denote the set of all nonnegative functions from $L_1(M)$.

Recall that a set of functions $G \subset L_1(M)$ is said to be *uniformly integrable* (on M) if, for any $\varepsilon > 0$, there exists a $\delta > 0$ satisfying the following condition: if a measurable set $E \subset M$ has $mes\,E < \delta$, then

$$\int_E |g(t)|\, dt < \varepsilon \qquad \text{for any} \quad g \in G.$$

Definition 1 (Balder [3]). A set of functions $G \subset L_1(M)$ is said to be *strongly uniformly integrable* if, for any $\varepsilon > 0$, there exists an $h \in L_1^+(M)$ such that

$$\int\limits_{E(|g|>h)} |g(t)|\, dt < \varepsilon \qquad \text{for any} \quad g \in G,$$

where $E(f > h) = \{\, t \in M \mid f(t) > h(t)\}$.

It is easy to see that this property coincides with the uniform integrability for a set M of a finite measure and it is stronger than the latter property for a set of an infinite measure.

Instead of Definition 1, we shall use the following equivalent Definition 2, which seems to be more convenient.

Let $h, g \in L_1^+(M)$. We shall say that h *majorizes* g if $h(t) \geq g(t)$ a.e. on M, and h *majorizes* g *with an integral accuracy* $\varepsilon > 0$ if

$$\int\limits_M (g(t) - h(t))^+\, dt < \varepsilon\, .$$

Definition 2. A set of functions $G \subset L_1(M)$ is said to be *strongly uniformly integrable* if, for any $\varepsilon > 0$, there exists a function $h \in L_1^+(M)$ majorizing $|g(t)|$ with the integral accuracy ε for any $g \in G$.

Lemma 11. *Definition 1 and Definition 2 are equivalent.*

Proof. Without loss of generality, we assume that $G \subset L_1^+(M)$. Since $h \geq 0$, we always have $g - h \leq g$ and hence

$$\int\limits_M (g-h)^+\, dt = \int\limits_{E(g>h)} (g-h)\, dt \ \leq \int\limits_{E(g>h)} g\ dt\ .$$

Therefore, any set G satisfying Definition 1 also satisfies Definition 2.

Let us prove the converse. Let a set G satisfy Definition 2. Assume that it does not satisfy Definition 1, i.e., there exists an $\varepsilon > 0$ such that, for any $h \in L_1^+(M)$, there is a $g_h \in G$ for which

$$\int\limits_{E_h} g_h\, dt \ \geq\ \varepsilon > 0, \qquad \text{where } E_h = E(g_h > h). \tag{13}$$

Fix this ε. By Definition 2, there exists an $h_0 \in L_1^+(M)$ such that the inequality

$$\int\limits_M (g - h_0)^+\, dt \ <\ \varepsilon/2$$

holds for any $g \in G$. Then we also have

$$\int\limits_M (g - h)^+\, dt \ <\ \varepsilon/2$$

for any $h \geq h_0$. Therefore, for any $h \geq h_0$, we have

$$\int\limits_M (g_h - h)^+ \, dt \;=\; \int\limits_{E_h} (g_h - h) \, dt \;<\; \varepsilon/2 \,,$$

and hence, with account of (13) we obtain

$$\int\limits_{E_h} h \, dt \;>\; \varepsilon/2 \,.$$

By definition, we always have

$$g_h - h/2 \;= (g_h - h) + h/2 \;> h/2 \qquad \text{on } E_h \,;$$

whence, in view of the preceding inequality, we obtain

$$\int\limits_{E_h} (g_h - h/2) \, dt \;>\; \varepsilon/4 \,,$$

and since $E(g_h > h/2) \supset E(g_h > h) = E_h$, we have

$$\int\limits_{E(g_h > h/2)} (g_h - h/2) \, dt \;=\; \int\limits_M (g_h - h/2)^+ \, dt \;>\; \varepsilon/4 \,.$$

This inequality holds for any $h \geq h_0$. But this contradicts Definition 2, according to which there exists an $h \geq h_0$ such that $\forall \, g \in G$

$$\int\limits_M (g - h/2)^+ \, dt \;<\; \varepsilon/4 \,.$$

The lemma is proved. $\square$

Consider the case $M = (0, +\infty)$. Let a set $\Phi \subset L_1(0, \infty)$ be such that the family of functions $\{\varphi^- \mid \varphi \in \Phi\}$ is strongly uniformly integrable. Using Def. 2 and the relation $a^- = (-a)^+$, one can easily show that this condition is equivalent to the following one: for any $\varepsilon > 0$, there is an $h_\varepsilon \in L_1^+(0, \infty)$ such that $\forall \, \varphi \in \Phi$

$$\int\limits_0^\infty (\varphi + h_\varepsilon)^- \, dt < \varepsilon.$$

Let us show that, in this case, our condition about the uniform convergence to zero of the negative part of the functional portions also holds for the set Φ. For any $T' < T''$, we have

$$\int\limits_{T'}^{T''} (\varphi + h_\varepsilon)^- \, dt \;\leq\; \int\limits_0^\infty (\varphi + h_\varepsilon)^- \, dt < \varepsilon.$$

Since h_ε is integrable, there exists a T_ε such that

$$\int_{T_\varepsilon}^{\infty} h_\varepsilon \, dt < \varepsilon.$$

Represent φ in the form $\varphi = (\varphi + h_\varepsilon) + (-h_\varepsilon)$. Since the function $(\cdot)^-$ is sublinear and $h_\varepsilon \geq 0$, we have $\varphi^- \leq (\varphi + h_\varepsilon)^- + (-h_\varepsilon)^- = (\varphi + h_\varepsilon)^- + h_\varepsilon$.

Then, for any $T'' > T' > T_\varepsilon$, we obtain

$$\int_{T'}^{T''} \varphi^- \, dt \;\leq\; \int_{T'}^{T''} (\varphi + h_\varepsilon)^- \, dt \;+ \int_{T'}^{T''} h_\varepsilon \, dt < \;\varepsilon + \varepsilon \;= 2\varepsilon,$$

whence, again by the sublinearity of the function $(\cdot)^-$, we get

$$\left(\int_{T'}^{T''} \varphi \, dt \right)^- \;\leq\; \int_{T'}^{T''} \varphi^- \, dt \;< 2\varepsilon\,,$$

which implies that Assumption A8 for the set Φ is also satisfied.

But the converse statement is obviously not true, even because the functions satisfying A8 may not be Lebesgue integrable on $(0, \infty)$ (i.e., their integrals may not converge absolutely). For example, if, instead of the functions $\varphi(t) \in \Phi$, we consider $\varphi(t) + \frac{1}{t+1} \sin t$, then the new functions would not belong to $L_1(0, \infty)$, while the portions of their integrals would still satisfy estimate (6), because

$$\int_{T'}^{T''} \frac{\sin t}{t+1} \, dt \;=\; -\frac{\cos t}{t+1} \bigg|_{T'}^{T''} \;- \int_{T'}^{T''} \frac{\cos t}{(t+1)^2} \, dt \;\to 0$$

as $T', T'' \to \infty$. Thus, the requirement imposed in [3] on the family of functions to be strongly uniformly integrable is more restrictive than our Assumption A8.

If one even weakens the condition of [3] by requiring that the family of functions $\{\varphi^-(t)\}$ be strongly uniformly integrable only on the interval (T, ∞) for a sufficiently large T, then, by the same reason, this weakened requirement remains to be strictly stronger than our Assumption A8.

7 Possible generalizations

1. If problem (1)–(5) contains additional inequality constraints of the form

$$J_i = \;\beta_i(x(0)) + \int_0^{\infty} \varphi_i(t, x, u) \, dt \leq 0, \qquad i = 1, \ldots, m, \tag{14}$$

where the functions β_i and φ_i satisfy the same assumptions as β and φ (also including Assumption A8) and the integral over $[0, \infty)$ is still understood as the limit of integrals over the intervals $[0, T]$ as $T \to \infty$, then it is clear that the main theorem remains true and the proof does not change. Indeed, the functionals J_i are lower semicontinuous on the "old" set Ω (which does not take constraint (14) into account) with respect to the convergence introduced above. Hence the set of pairs $(x, u) \in \Omega$ satisfying constraint (14) is a closed set, and then the new set of admissible trajectories, which is now determined by constraints (2)–(5) and (14), is still compact.

2. The problem can also contain additional equality constraints of the form

$$F_j = \beta_j(x(0)) + \int_0^\infty f_j(t, x, u)\, dt = 0, \qquad j = 1, \ldots, k, \tag{15}$$

where the functions f_j satisfy Assumption A1 (i.e., they are continuous in (t, x) and linear in u) and the β_j satisfy Assumption A6 (i.e., they are continuous). In this case, we impose the requirement that the functions f_j and $-f_j$ satisfy Assumption A8, i.e., that

$$\int_{T'}^{T''} f_j(t, x(t), u(t))\, dt \to 0 \qquad \text{as} \qquad T', T'' \to \infty$$

uniformly over the "old" set Ω. The last condition is equivalent to the condition that, for any admissible pair, the integrals in (15) have finite values and, moreover, converge to their values uniformly over all admissible trajectories. In this situation, the main theorem is again true, since the set of pairs $(x, u) \in \Omega$ satisfying (15) is closed in the introduced convergence.

Constraint (15) allows one to consider final equality constraints on the endpoint of the trajectory $x(\infty) = \lim_{T \to \infty} x(T)$ of the form

$$(a_j, x(\infty)) = c_j, \qquad j = 1, \ldots, k,$$

where $a_j \in \mathbb{R}^n$, which can equivalently be written in the form

$$(a_j, x(0)) + \int_0^\infty (a_j, f(t, x(t), u(t)))\, dt = c_j, \qquad j = 1, \ldots, k\,.$$

Here one should assume that the functions $\pm(a_j, f(t, x, u))$ satisfy Assumption A8.

3. The problem containing, instead of the controlled system, a differential inclusion $\dot{x} \in V(t, x)$, where the set-valued mapping V satisfies Assumption A4, can

easily be reduced to the problem considered above. To do this, one should pass to the system $\dot{x} = u, \ \ u \in V(t,x)$.

4. In the present paper we assume that system (2) is linear in the control, and the set $U(t,x)$ is convex and compact. In the case of general nonlinear system, one should require that this set be bounded, while convex and closed be the velocity set of extended system [10, 3]

$$\dot{y} = \varphi(t,x,u) + v, \quad \dot{x} = f(t,x,u), \qquad u \in U(t,x), \quad v \ge 0.$$

In this case, one should consider the convergence of trajectories $(y(t),\, x(t))$ in the space $C[0,T] \times C^n[0,T]$ and to apply a certain version of the measurable selection theorem, e.g., Filippov's inclusion lemma [8], in order to represent the limit trajectory as a solution to system (2) for some control $u(t)$.

Here we do not consider this general case, because the corresponding technical complications concern the problem on a fixed interval (and are well known), while our goal is to show, as clearly as possible, the specificity of the problem on infinite interval. Note only, that this general case completely falls into the abstract scheme proposed below.

8 Abstract scheme

The above method of the proof also remains valid in the following abstract setting.

Let be given an increasing countable family of sets $T_n \subset T_{n+1} \subset ...,\ \ n = 1, 2, ...,$ and $\Re = \bigcup T_n$. On each T_n, there is a set of functions $\Omega(T_n) = \{\, w : T_n \to Z \,\}$ ranging in some set Z of images, and a functional $J_n :\ \Omega(T_n) \to \mathbb{R}$. Assume that for any n we have $\Omega(T_{n+1})|_{T_n} \subset \Omega(T_n)$.

Let Ω be the set of all functions $w : \Re \to Z$ such that, for any n, the restriction $w_n = w\,|_{T_n}$ belongs to $\Omega(T_n)$. Then, for each function $w \in \Omega$ one can define the functionals $J_n(w) = J_n(w_n),\ n = 1, 2, \ldots,$ and hence one can consider the functional

$$J(w) = \lim_{n\to\infty} J_n(w_n).$$

More precisely, this functional is considered only for those $w \in \Omega$ for which this limit (finite or infinite) exists.

Now, we pose the problem: $J(w) \to min$ over all $w \in \Omega$ for which the functional exists.

Assumptions.

S1. For each n, the set of functions $\Omega(T_n)$ is compact in a certain topology τ_n having a countable base (and hence metrizable); moreover, the mapping π_n :

$\Omega(T_{n+1}) \to \Omega(T_n)$ associating to each function $w(t)$ on the set T_{n+1} its restriction to T_n is continuous.

S2. For each n, the functional J_n is lower semicontinuous on $\Omega(T_n)$ in the topology τ_n .

S3. There exists a number sequence $\alpha_N \to 0+$ and a number N_0 such that, for any $N > N_0$ and any $n_2 > n_1 \geq N$, the inequality

$$J_{n_2}(w) - J_{n_1}(w) \ \geq \ -\alpha_N$$

holds for any $w \in \Omega$ or, which is the same,

$$(J_{n_2}(w) - J_{n_1}(w))^- \to 0 \qquad \text{as} \quad n_1, n_2 \to \infty \,, \quad n_1 < n_2 \,,$$

uniformly over all $w \in \Omega$.

Thus, the differences $J_{n_2}(w) - J_{n_1}(w)$ play the role of the functional portions in this abstract scheme.

Under the above assumptions, an analog of Lemma 2 is valid, which guarantees the existence of the limit of the functional for any $w \in \Omega$ (as in above, this limit is either finite or equal to $+\infty$), the set Ω is metrizable in the topology of convergence on each T_n , the functional J is lower semicontinuous on Ω with respect to this convergence, and the following assertion holds.

Theorem 2. *Suppose that there exists a* $w \in \Omega$ *for which* $J(w) < +\infty$. *Then the above problem has a solution, i.e.,* J *attains its minimum on* Ω .

Proof. The proof repeats the main steps of the proof of Theorem 1. □

It is also possible to propose a yet more abstract scheme in which there are no sets T_n , the function spaces $\Omega(T_n)$ are replaced by an arbitrary projective family of compact sets Ω_n with countable bases, and Ω is the projective limit of this family. But for now, there are no convincing motivations to study the problem in this setting (and, moreover, to avoid the assumption that the topologies τ_n have countable bases); so, we do not consider it in detail here.

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